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GLOBAL WEATHER MODELING WITH VECTOR SPHERICAL HARMONICS

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I. INTRODUCTION

This report presents the current status of the work under ARPA order number 2089 , dealing with weather modeling with a view toward climatological applications. Emphasis is on the solution of the primitive equations governing weather by means of Vector Spherical Harmonics. (VSH) This section presents a summary of the report and a brief review of vector spherical harmonics.

Section II deals with theoretical developments of various aspects of the technique, notably the "Balance" conditions, advection formulas, kinetic energy, and an analysis of sound waves, all from the VSH viewpoint.

Section III deals with the modeling activities proper, including the derivation of a set of equations suitable for VSH solution and their subsequent implementation.

Section IV is a review of computational considerations mostly due to the transform method of handling nonlinear terms.

Section V discusses radial functions, i.e., the functions used to represent the radial dependence of the VSH coefficients.

Section VI reviews the results of fitting VSH coefficients to published data, and Section VII discusses the current status of the programming system.

Equation numbers are repeated between sections. All references to equations are within the same section unless otherwise indicated.

I.1 Brief Review of Vector Spherical Harmonics

The Vector Spherical Harmonics are a set of orthonormal functions of the spherical coordinates θ (colatitude) and φ (longitude). They are triply indexed functions denoted by

$$T_{JL}^{M}(\theta,\varphi) \tag{1}$$

where

$$J, L \geq 0, \quad |M| \leq L, \quad |J-L| \leq 1$$

a vector field in spherical coordinates may be expanded in a series of T_{JL}^{M}'s. If the vector field $\bar{V}$ is a function of the radius, θ, and φ, then

$$\bar{V}(r,\theta,\varphi) = \sum_{MJL} V_{JL}^{M}(r)\, T_{JL}^{M}(\theta,\varphi) \tag{2a}$$

$$V_{JL}^{M}(r) = \int_0^{2\pi}\int_0^{\pi} f(r,\theta,\varphi)\, Y_L^{M*}(\theta,\varphi) \sin\theta \, d\theta \, d\varphi \tag{2b}$$

If the vector field is a function of time as well, the coefficients V_{JL}^{M} are a function of r and t. To define the T_{JL}^{M}, we define first the ordinary scalar spherical harmonics (SSH) by

$$Y_L^M(\theta,\varphi) = (-1)^M \left[\frac{(2L+1)(L-M)!}{4\pi(L+M)!}\right]^{1/2} P_L^M(\cos\theta)e^{iM\varphi} \tag{3}$$

where $P_L^M(\cos\theta)$ are associated Legendre functions. Define a set of Basis Vectors $\hat{e}_{-1}$, $\hat{e}_0$, $\hat{e}_{+1}$ by

$$\hat{e}_{-1} \equiv \hat{e}_- = \frac{1}{\sqrt{2}}(\hat{i} - i\hat{j}) \tag{4}$$

$$\hat{e}_0 = \hat{k} \tag{5}$$

$$\hat{e}_{+1} \equiv \hat{e}_+ = \frac{1}{\sqrt{2}}(\hat{i} + i\hat{j}) \tag{6}$$

where, $\hat{i}$, $\hat{j}$, $\hat{k}$ are the usual cartesian base vectors and $i = \sqrt{-1}$.

The vector spherical harmonics are defined by the following expression:

$$T_{JL}^M(\theta,\varphi) = \sum_{\mu=-1}^{+1} \langle L\ M-\mu\ 1\ \mu | J\ M\rangle Y_L^{M-\mu}(\theta,\varphi)\hat{e}_\mu \tag{7}$$

The quantity $\langle L\, M-\mu\, 1\, \mu \mid J\, M \rangle$ is called a Clebsch-Cordon coefficient. It is a number depending on L, M, and μ. The properties of these coefficients and of $Y_L^M(\theta,\varphi)$ restrict L, J, and M to integer values such that $J \geq 0$; $L \geq 0$, and L = J-1, J, J+1; and $|M| \leq J$. There are then three sets of functions which we label

$$T_{L\,L-1}^{M}(\theta,\varphi) \qquad T_{L\,L}^{M}(\theta,\varphi) \qquad T_{L\,L+1}^{M}(\theta,\varphi) \tag{8}$$

The orthonormality properties of the T_{JL}^{M} are expressed by

$$\int_0^{2\pi}\int_0^{\pi} T_{JL}^{M\,*}(\theta,\varphi)\cdot T_{J'L'}^{M'}(\theta,\varphi)\sin\theta\, d\theta\, d\varphi = \delta_{MM'}\,\delta_{LL'}\,\delta_{JJ'} \tag{9}$$

Where * denotes complex conjugation and δ_{ij} is the Kroeneker delta.

The differential vector operations satisfied by the T_{JL}^{M} are given below. Note that no derivatives with respect to θ, φ appear. Let f(r) be an arbitrary function of r and

$$\alpha_L \equiv \sqrt{\frac{L+1}{2L+1}}$$

$$\beta_L \equiv \sqrt{\frac{L}{L+1}} \tag{10}$$

$$D_L \equiv \left(\frac{\partial}{\partial r} + \frac{L}{r}\right)$$

Gradient:

$$\nabla f(r) Y_L^m(\theta,\phi) = \beta_L D_{L+1} f(r) T_{LL-1}^m(\theta,\phi) - \alpha_L D_{-L} f(r) T_{LL+1}^m(\theta,\phi) \tag{11}$$

Divergence:

$$\nabla \cdot f(r) T_{LL-1}^M(\theta,\phi) = \beta_L D_{1-L} f(r) Y_L^M(\theta,\phi) \tag{12}$$

$$\nabla \cdot f(r) T_{LL}^M(\theta,\phi) = 0 \tag{13}$$

$$\nabla \cdot f(r) T_{LL+1}^M(\theta,\phi) = -\alpha_L D_{L+2} f(r) Y_L^M(\theta,\phi) \tag{14}$$

Curl:

$$\nabla \times f(r) T_{LL-1}^M(\theta,\phi) = i\alpha_L D_{1-L} f(r) T_{LL}^m \tag{15}$$

$$\nabla \times f(r) T_{LL}^m(\theta,\phi) = i\alpha_L D_{L+1} f(r) T_{LL-1}^m - i\beta_L D_{-L} f(r) T_{LL+1}^m \tag{16}$$

$$\nabla \times f(r)\, T^{m}_{L L+1}(\theta,\varphi) = i \beta_L D_{L+2} f(r)\, T^{m}_{LL} \tag{17}$$

Product-type nonlinearities are handled by the well-known formula for scalar spherical harmonics:

$$Y_L^M Y_{L'}^{M'} = \frac{1}{\sqrt{4\pi}} (2L+1)(2L'+1) \sum_{L''=|L-L'|}^{L+L'} \langle L\,0\,L'\,0 | L''\,0 \rangle \tag{18}$$

$$\langle L\,M\,L'\,M' | L''\,M+M' \rangle Y_{L''}^{M+M'}$$

For the term $(\bar{V}\cdot\nabla)\bar{V}$, which occurs in Hydrodynamics and other fields ($\bar{V}$ is a vector field), the expression (apart from normalizing coefficients):

$$\left(f_1(r)\, T^{m}_{JL}(\theta,\varphi)\cdot\nabla \right) f_2(r)\, T^{m'}_{J'L'} = (\text{constants})\cdot f_1(r) \tag{19}$$

$$\sum_{L''J''} W(L'J'L''J'';1J) \langle J\,M\,J'\,M' | J''(M+M') \rangle$$

$$\Big\{ Z(LJL'-1L';1L'')\,\beta_{L'} D_{L'+1} +$$

$$Z(LJL'+1L';1L'')\,\alpha_{L'} D_{-L'} \Big\} f_2(r)\, T^{M+M'}_{L''}$$

For many applications it is convenient to use linear combinations of T_{JL}^{M}'s which are closer to the physical coordinate system. Such a set is given by the set $A_L^M(\theta,\phi)$, $B_L^M(\theta,\phi)$, and $C_L^M(\theta,\phi)$, defined by

$$A_L^M(\theta,\phi) = \beta_L T_{LL-1}^{M}(\theta,\phi) - \alpha_L T_{LL+1}^{M}(\theta,\phi) \tag{20}$$

$$B_L^M(\theta,\phi) = \alpha_L T_{LL-1}^{M}(\theta,\phi) + \beta_L T_{LL+1}^{M}(\theta,\phi) \tag{21}$$

$$C_L^M(\theta,\phi) = T_{LL}^{M}(\theta,\phi) \tag{22}$$

It may also be shown that

$$A_L^M(\theta,\phi) = Y_L^M(\theta,\phi)\hat{e}_r \tag{23}$$

$$B_L^M(\theta,\phi) = \frac{r}{\gamma_L}\nabla Y_L^M \tag{24}$$

$$C_L^M(\theta,\phi) = \frac{-ir}{\gamma_L}\hat{e}_r \times \nabla Y_L^M \tag{25}$$

Where

$$\gamma_L = \sqrt{L(L+1)} \tag{26}$$

From the above equations, the $A_L^m(\theta,\varphi)$ are along the radial direction, while $B_L^m(\theta,\varphi)$ and $C_L^m(\theta,\varphi)$ are tangential to the surface of the earth. For the case $m=0$, $B_L^0(\theta)$ is meridional and $C_L^0(\theta)$ is zonal.

In analogy with equation (2a), the vector field $\overline{V}$ can be expanded as follows:

$$\overline{V}(r,\theta,\varphi) = \sum \left[a_L^m(r) A_L^m(\theta,\varphi) + b_L^m(r) B_L^m(\theta,\varphi) + c_L^m(r) C_L^m(\theta,\varphi) \right] \tag{27}$$

The spatial derivative operations in this representation are

Gradient:

$$\nabla f(r) Y_L^m = D_0 f(r) A_L^m + \frac{\gamma_L}{r} f(r) B_L^m \tag{28}$$

Divergence:

$$\nabla \cdot f(r) A_L^m = D_2 f(r) Y_L^m \tag{29}$$

$$\nabla \cdot f(r) B_L^m = -\frac{\gamma_L}{r} f(r) Y_L^m \tag{30}$$

$$\nabla \cdot f(r) C_L^m = 0$$

(31)

Curl:

$$\nabla \times f(r) A_L^m = -\frac{i \gamma_L}{r} f(r) C_L^m$$

(32)

$$\nabla \times f(r) B_L^m = i D_1 f(r) C_L^m$$

(33)

$$\nabla \times f(r) C_L^m = \frac{i \gamma_L}{r} f(r) A_L^m + i D_1 f(r) B_L^m$$

(34)

Here, and usually in the following equations, we will omit arguments on the functions.

These results are discussed in previous reports in considerable detail. The advantage of VSH in solving partial differential equations is that the angular part of the equation can be separated out using (28)-(34); reducing the equations to partial differential equations in the variables r and t. By expanding the coefficients in suitable functions of r, the equations can be further reduced to ordinary differential equations in time. This is made possible by the fact that none of the spatial differential operations (curl, divergence, gradient, etc.) involve derivatives with respect to θ or φ, and by the orthonormality of the Vector Spherical Harmonics.

Computationally, it is advantageous to use the Transform method to evaluate nonlinear terms (see report for previous year). In this method, the fields are converted from spectral to physical form by evaluating equation (2a) or its scalar analog for each variable in the nonlinear term. The result is the variables represented on a physical grid. The nonlinear operation is then performed on the physical grid and the result converted back to coefficients by equation (2b). In practice, this saves considerable amounts of computation.

II.1 Balance Conditions and Truncated Spectral Forms

The purpose of the following sections is to investigate the various meteorological "balance" conditions when the equations are cast in spectral form. As will appear, great care must be exercised in order to avoid inconsistencies or anomalous results.

II.1.1 The Hydrostatic Balance Equation

The hydrostatic balance equation is

$$\frac{\partial p}{\partial r} = -\rho g \quad \text{or} \quad \alpha \frac{\partial p}{\partial r} = -g \tag{1.1}$$

To illustrate the difficulty, suppose we expand $\partial_r p$ and α in scalar spherical harmonics with only two terms (to simplify the algebra), then

$$\alpha = \alpha_0 Y_0^o + \alpha_1^o Y_1^o \tag{1.2}$$

$$\frac{\partial p}{\partial r} = \beta_0^o Y_0^o + \beta_1^o Y_1^o \tag{1.3}$$

the product is then

$$\alpha \frac{\partial p}{\partial r} = \alpha_0^o \beta_0^o Y_0^o + \alpha_0^o \beta_1^o Y_0^o Y_1^o + \alpha_1^o \beta_0^o Y_1^o Y_0^o + \alpha_1^o \beta_1^o Y_1^o Y_1^o \tag{1.4}$$

Now by the product rules

$$\begin{aligned} Y_0^o Y_0^o &= C_0 Y_0^o \\ Y_1^o Y_0^o &= C_1 Y_1^o \\ Y_1^o Y_1^o &= C_2 Y_0^o + C_3 Y_2^o \end{aligned} \tag{1.5}$$

and
so the equation (1.1) is then

$$\alpha_0^o \beta_0^o C_0 Y_0^o + (\alpha_0^o \beta_1^o + \alpha_1^o \beta_0^o) C_1 Y_1^o + \alpha_1^o \beta_1^o (C_2 Y_0^o + C_3 Y_2^o) = g_0^o Y_0^o \tag{1.6}$$

or, collecting coefficients and equating:

$$\alpha_0^0 \beta_0^0 C_0 + \alpha_1^0 \beta_1^0 C_2 = g_0^0 \tag{1.7}$$

$$C_1 \alpha_0^0 \beta_1^0 + C_1 \alpha_1^0 \beta_0^0 = 0 \tag{1.8}$$

$$\alpha_1^0 \beta_1^0 = 0 \tag{1.9}$$

Now in order to satisfy (1.8), either α_1^0 or β_1^0 must be zero (or both). If α_1^0 is zero, then (1.7) forces either α_0^0 or β_1^0 to be zero. α_0^0 cannot be zero since in (1.7), $g_0^0 \neq 0$, hence $\alpha_1^0 = 0$. This leaves us with the equation

$$\alpha_0^0 \beta_0^0 = g_0^0 \tag{1.10}$$

If g is independent of r, as usually approximated, then the product of α_0^0 and β_0^0 must be independent of r; that is

$$\frac{\partial}{\partial r}(\alpha_0^0 \beta_0^0) = 0 \tag{1.11}$$

or, substituting $\beta_0^0 = \frac{\partial \rho_0^0}{\partial r}$ we have from 1.11

$$\frac{\partial \alpha_0^0}{\partial r} \frac{\partial \rho_0^0}{\partial r} + \alpha_0^0 \frac{\partial^2 \rho_0^0}{\partial r^2} = 0 \tag{1.12}$$

Thus we are left to infer that if g is independent of θ and φ, and α and ρ are represented by truncated series, then α and ρ are independent of θ and φ. Furthermore, the coefficients of r and p are related by (1.11).

Although we were led to this conclusion by an example, and not by a formal demonstration, the difficulty will persist as we take more terms, i.e., pressure and density will still be independent of θ and φ.

It is only with an infinite number of terms that (1.1) can be satisfied at all (because then g/t has an expansion). In fact, of course, α and ρ are dependent on θ and φ. This points up the fact that constraint (1.1) is very difficult to satisfy in spectral form, at least with finite expansions. If g is allowed to be a function of θ and φ, the difficulty will be removed up to the number of terms in which g is expanded.

The main problem arising with the preceding equation is the fact that we have not been consistent regarding the order of expansion. Thus, while we started out with expansions of order $L=1$, we retained terms in the product of order $L=2$. When dealing with truncated expansions, it is essential to truncate the product at the original level. When we do this, equation (1.6) becomes

$$\alpha_0^0 \beta_0^0 c_0 Y_0^0 + (\alpha_0^0 \beta_1^0 + \alpha_1^0 \beta_0^0) c_1 Y_1^0 + \alpha_1^0 \beta_1^0 c_2 Y_0^0 = g_0^0 Y_0^0 \tag{1.13}$$

Now, collecting coefficients leads to the equations

$$\alpha_0^0 \beta_0^0 c_0 + \alpha_1^0 \beta_1^0 c_2 = g_0^0 \tag{1.14}$$

$$\alpha_0^0 \beta_1^0 c_1 + \alpha_1^0 \beta_0^0 c_2 = 0 \tag{1.15}$$

So that now, if (say) α_L^0 are given, we can solve the system (1.14) and (1.15) for ρ_0^0 and β_1^0. In this sense, equations such as (1.6) are consistent, in the sense that given the coefficients of either α or $\partial\rho/\partial r$, we can solve a system of equations for the coefficients of the other. The system will be consistent only if care is exercised with the truncation limits when the product is formed.

II.1.2 Geostrophic Balance Condition

The geostrophic balance condition assumes, in cartesian coordinates, that

$$f v = \frac{1}{\rho}\frac{\partial p}{\partial x} \tag{1.16}$$

$$f u = -\frac{1}{\rho}\frac{\partial p}{\partial y} \tag{1.17}$$

where f is the coriolis parameter, and u and v horizontal wind components. In vector form,

$$(2\Omega \hat{k} \times \bar{V}_H)_H = \frac{1}{\rho} \nabla_H p \tag{1.18}$$

Since this equation involves products, it will be convenient to use pressure coordinates, in which case assuming the hydrostatic balance condition $\partial_r p = -\rho g$ the equation becomes ($\bar{V}_H$ is now a function of p, θ and φ).

$$(2\Omega \hat{k} \times \bar{V}_H)_H = -g \nabla_p r \equiv -\nabla_p \phi \tag{1.19}$$

A discussion of pressure coordinates is contained in Section 3.5. The equation is now linear. We can immediately conclude that since the gradient operation ∇_p only produces B_L^n vector spherical harmonics, the only terms in $\bar{V}_H$ that can be determined are those in $\hat{k} \times \bar{V}_H$ producing B_L^M's. Since the expression for $\hat{k} \times \bar{V}$ is cumbersome, we will first discuss the case $M=0$ (no φ dependence). In this case, the expressions for $\hat{k} \times \bar{V}$ are:

$$\hat{k}\times b_L^0 B_L^0 = -i\left(\frac{\alpha_L\beta_{L-1}}{\beta_L}\langle L010|L-1\,0\rangle b_L^0 C_{L-1}^0 - \frac{\beta_L\alpha_{L-1}}{\alpha_L}\langle L010|L+1\,0\rangle b_L^0 B_{L+1}^0\right) \tag{1.20}$$

$$\hat{k}\times c_L^0 C_L^0 = -i\left(\frac{\alpha_L\beta_{L-1}}{\beta_L}\langle L010|L-1\,0\rangle c_L^0 B_{L-1}^0 - \frac{\beta_L\alpha_{L+1}}{\alpha_L}\langle L010|L+1\,0\rangle c_L^0 B_{L+1}^0\right) \tag{1.21}$$

From (1.20) and (1.21) we see that only the terms in $\bar{V}$ of the type $c_L^0 C_L^0$ will produce B_L^0 terms. In order to see the structure of the resulting terms, it is convenient to write equations (1.20) and (1.21) in matrix form. The coefficients will be denoted by α_i, as their form is not important. The equations are carried out to $L=5$ for purposes of illustration.

$$\hat{k}\times\bar{V} = \hat{k}\times\left[\, c_1^0 \;\; c_2^0 \;\; c_3^0 \;\; c_4^0 \;\; c_5^0 \;\; b_1^0 \;\; b_2^0 \;\; b_3^0 \;\; b_4^0 \;\; b_5^0 \,\right]\begin{bmatrix} C_1^0 \\ C_2^0 \\ C_3^0 \\ C_4^0 \\ C_5^0 \\ B_1^0 \\ B_2^0 \\ B_3^0 \\ B_4^0 \\ B_5^0 \end{bmatrix} \tag{1.22}$$

$$=$$

$$\begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array}$$
$$\begin{bmatrix} 0 & \alpha_2 c_1^0 & 0 & 0 & 0 & 0 & \alpha_{10} b_1^0 & 0 & 0 & 0 \\ \alpha_1 c_2^0 & 0 & \alpha_4 c_2^0 & 0 & 0 & \alpha_9 b_2^0 & 0 & \alpha_{12} b_2^0 & 0 & 0 \\ 0 & \alpha_3 c_3^0 & 0 & \alpha_6 c_3^0 & 0 & 0 & \alpha_{11} b_3^0 & 0 & \alpha_{14} b_3^0 & 0 \\ 0 & 0 & \alpha_5 c_4^0 & 0 & \alpha_8 c_4^0 & 0 & 0 & \alpha_{13} b_4^0 & 0 & \alpha_{16} b_4^0 \\ 0 & 0 & 0 & \alpha_7 c_5^0 & 0 & 0 & 0 & 0 & \alpha_{15} b_5^0 & 0 \end{bmatrix}\begin{bmatrix} B_1^0 \\ B_2^0 \\ B_3^0 \\ B_4^0 \\ B_5^0 \\ C_1^0 \\ C_2^0 \\ C_3^0 \\ C_4^0 \\ C_5^0 \end{bmatrix} \tag{1.23}$$

Since on the right of equation (1.19) we have $\nabla_p r$, we will have terms on the right of the form $Y_L r_1^M B_L^M / p$, where r_L^M are the coefficients of r. Then upon collecting coefficients of B_L^M and C_L^M on both sides of (1.19) and equating, the following system of equations is obtained, where $k_L = \frac{\delta_L g}{2r\Omega}$:

$$
\begin{aligned}
\alpha_1 c_2^0 &= k_1 r_1^0 \\
\alpha_2 c_1^0 + \alpha_3 c_3^0 &= k_2 r_2^0 \\
\alpha_4 c_2^0 + \alpha_5 c_4^0 &= k_3 r_3^0 \\
\alpha_6 c_3^0 + \alpha_7 c_5^0 &= k_4 r_4^0 \\
\alpha_8 c_4^0 &= k_5 r_5^0 \\
\alpha_9 b_2^0 &= 0 \\
\alpha_{10} b_1^0 + \alpha_{11} b_3^0 &= 0 \\
\alpha_{12} b_2^0 + \alpha_{13} b_4^0 &= 0 \\
\alpha_{14} b_3^0 + \alpha_{15} b_5^0 &= 0 \\
\alpha_{16} b_4^0 &= 0
\end{aligned}
\qquad (1.24)
$$

As we can see, this system does not couple b_L^0 with c_L^0 and the two can be determined separately, if they can be determined at all. We immediately observe some major difficulties. Considering first the equations for the b_L^0, eq., (1.24.6) forces $b_2^0 = 0$. Then from equation (1.24.8) $b_4^0 = 0$, and so forth. Thus, the even-L $b_L^0 = b_{2k}^0$ are zero. But the odd-L b_L^0 cannot be determined at all, as we have:

$$\alpha_{10} b_1^0 + \alpha_{11} b_3^0 = 0$$

$$\alpha_{14} b_3^0 + \alpha_{15} b_5^0 = 0$$

which is too few equations. If we had truncated at an even-order L we could use the last equation to infer that the highest-order odd-L b_L^0 is zero, but this is quite arbitrary.

In similar fashion, for the c_L^0, eq. (1.24.1) yields c_2^0, using this value in (1.24.3) yields c_4^0 and so forth. The result of truncating

at $L=5$ yields (1.24.5) which gives a contradictory value for C_4^o. In analogy to the above discussion for the b_L^o, the C_L^o for odd L are not determinable from the above in this example. Extending the example will remove the problem for the odd L, using the last equation of the set. But if we truncate at odd L we must in fact presumably neglect this equation.

All these results point up the inadequacy of the geostrophic balance condition in spectral form. The reason for the inadequacy lies in the fact that geostrophic balance holds only at middle latitudes. Spectral forms, however, are global, and an approximation that holds only at restricted latitudes must necessarily present problems in a global form. To further elucidate the point, we can transform equations (1.19) to spherical coordinates, yielding:

$$-2\Omega \cos\theta \, V_\varphi = \frac{g}{r(p)} \frac{\partial r}{\partial \theta} \qquad (1.25)$$

$$2\Omega \cos\vartheta \, V_\vartheta = \frac{g}{r(p)} \frac{\partial r}{\partial \varphi} \qquad (1.26)$$

Now when M = 0, r is independent of φ and therefore $V_\vartheta = 0$. So we need only consider (1.25). The factor ($\cos\vartheta$) causes V_φ to approach infinity at the equator, unless $\partial r/\partial\vartheta$ also contains a $\cos\vartheta$ factor. In a spectral representation, $\partial r/\partial\vartheta$ contains associated Legendre polynomials, which are alternately zero and non-zero at the equator. Those that are non-zero at the equator cannot satisfy (1.25). Thus, (1.25) implies that only even-order coefficients C_{2K}^o can be determined. This is unfortunate, since data analysis shows that the odd-ordered C_L^o are the major part of the zonal wind. (The zonal wind is in fact given by C_L^o.)

The preceding discussion points out the fact that the geostrophic balance is not a suitable approximation to make on any spectral model, at least one utilizing vector spherical harmonics.

Next, we consider the case where M $\neq$ 0, that is, when the fields r and ∇ depend on ϕ. In this case the formulas are:

$$\hat{k} \times b_L^M B_L^M = -i\left(G_- b_L^M C_{L-1}^M + G_0 b_L^M B_L^M + G_+ b_L^M C_{L+1}^M \right) \quad (1.27)$$

$$\hat{k} \times c_L^M C_L^M = -i\left(G_- c_L^M B_{L-1}^M + G_0 c_L^M C_L^M + G_+ c_L^M B_{L+1}^M \right) \quad (1.28)$$

where

$$G_- = \frac{\alpha_L \beta_{L-1}}{\beta_L} \langle L M 1 0 | L-1\, M \rangle$$

$$G_0 = \frac{1}{\gamma_L} \langle L M 1 0 | L M \rangle$$

$$G_+ = \frac{\beta_L \alpha_{L+1}}{\alpha_L} \langle L M 1 0 | L+1\, M \rangle$$

Let us consider the case $L_{MAX} = M_{MAX} = 3$ for purposes of illustration. We obtain from the equation (1.19) the expansion

$$2\Omega \hat{k} \times \Big(\sum_{L,M} b_L^M B_L^M + c_L^M C_L^M \Big) = -g \nabla_p r = -g \sum \frac{\gamma_L}{r(p)} r_L^M B_L^M \quad (1.29)$$

In analogy with the set of equations (1.23) we now have the array of coefficients on the left as follows: (we have omitted the $G_{+,0,-}$ factors for clarity)

$$
K \times \bar{V} \rightarrow
\begin{array}{r}
1\\2\\3\\4\\5\\6\\7\\8\\9\\10\\11\\12\\13\\14\\15\\16\\17\\18\\19\\20\\21\\22\\23\\24
\end{array}
\begin{bmatrix}
b_1^1 &&&&& &&&&& &&& b_1^1 &&&&& &&&&& \\
& b_1^{-1} &&&&& &&&&& &&& b_1^{-1} &&&&& &&&& \\
&& b_2^1 &&&&& &&&& b_2^1 &&&& b_2^1 &&&&& &&& \\
&&& b_2^{-1} &&&&& &&&& b_2^{-1} &&&& b_2^{-1} &&&&& && \\
&&&& b_3^1 &&&&& &&&& b_3^1 &&&&& &&&&& \\
&&&&& b_3^{-1} &&&&& &&&& b_3^{-1} &&&&& &&&& \\
&&&&& & b_2^2 &&&&& &&&&& &&& b_2^2 &&&& \\
&&&&& && b_2^{-2} &&&&& &&&&& &&& b_2^{-2} &&& \\
&&&&& &&& b_3^2 &&&&& &&&& b_3^2 &&&&& & \\
&&&&& &&&& b_3^{-2} &&&&& &&&& b_3^{-2} &&&&& \\
&&&&& &&&&& b_3^3 &&&&& &&&&& &&& \\
&&&&& &&&&& & b_3^{-3} &&&&& &&&&& && \\
&& c_1^1 &&&&& &&&&& c_1^1 &&&&& &&&&& & \\
&&& c_1^{-1} &&&&& &&&&& c_1^{-1} &&&&& &&&&& \\
c_2^1 &&&& c_2^1 &&&&& &&&&& c_2^1 &&&&& &&&& \\
& c_2^{-1} &&&& c_2^{-1} &&&&& &&&&& c_2^{-1} &&&&& &&& \\
&& c_3^1 &&&&& &&&&& &&&& c_3^1 &&&&& && \\
&&& c_3^{-1} &&&&& &&&&& &&&& c_3^{-1} &&&&& & \\
&&&&& &&& c_2^2 &&&&& &&&&& c_2^2 &&&&& \\
&&&&& &&&& c_2^{-2} &&&&& &&&&& c_2^{-2} &&&& \\
&&&&& & c_3^2 &&&&& &&&&& &&&& c_3^2 &&& \\
&&&&& && c_3^{-2} &&&&& &&&&& &&&& c_3^{-2} && \\
&&&&& &&&&& &&&&& &&&&& && c_3^3 & \\
&&&&& &&&&& &&&&& &&&&& &&& c_3^{-3}
\end{bmatrix}
\begin{bmatrix}
B_1^1\\ B_1^{-1}\\ B_2^1\\ B_2^{-1}\\ B_3^1\\ B_3^{-1}\\ B_2^2\\ B_2^{-2}\\ B_3^2\\ B_3^{-2}\\ B_3^3\\ B_3^{-3}\\ C_1^1\\ C_1^{-1}\\ C_2^1\\ C_2^{-1}\\ C_3^1\\ C_3^{-1}\\ C_2^2\\ C_2^{-2}\\ C_3^2\\ C_3^{-2}\\ C_3^3\\ C_3^{-3}
\end{bmatrix}
\qquad (1\text{-}30)
$$

On the right we have the terms containing B_L^M. Upon equating coefficients of the B_L^M and C_L^M in equation (1.28) we can write the following systems of equations:

$$\begin{cases} \alpha_1 b_1^1 + \alpha_2 c_2^1 = r_1^1 \\ \alpha_3 b_3^1 + \alpha_4 c_2^1 = r_3^1 \\ \alpha_5 b_1^1 + \alpha_6 b_3^1 + \alpha_7 c_2^1 = 0 \end{cases} \tag{1.31}$$

$$\begin{cases} \alpha_8 b_3^{-1} + \alpha_9 c_2^{-1} = r_3^{-1} \\ \alpha_{10} b_3^{-1} + \alpha_{11} c_2^{-1} = 0 \end{cases} \tag{1.32}$$

$$\begin{cases} \alpha_{12} b_2^2 + \alpha_{13} c_3^2 = r_2^2 \\ \alpha_{14} b_2^2 + \alpha_{15} c_3^2 = 0 \end{cases} \tag{1.33}$$

$$\begin{cases} \alpha_{16} b_3^2 + \alpha_{17} c_2^2 = r_3^2 \\ \alpha_{18} \cdot b_3^2 + \alpha_{19} c_2^2 = 0 \end{cases} \tag{1.34}$$

$$\begin{cases} \alpha_{20} b_2^1 + \alpha_{21} c_1^1 + \alpha_{22} c_3^1 = 0 \\ \alpha_{23} b_2^1 + \alpha_{24} c_3^1 = 0 \\ \alpha_{25} b_{12} + \alpha_{26} c_1^1 = 0 \end{cases} \tag{1.35}$$

$$\begin{cases} \alpha_{27} b_3^3 = r_3^3 \\ \alpha_{28} c_3^3 = 0 \end{cases} \tag{1.36}$$

where the α_i are coefficients of the form

$$\alpha_i = \frac{-g\gamma_L}{G_\mu 2\Omega r(p)}$$

and a similar set of equations can be written for negative M .
Thus we see that for $M \neq 0$ the geostrophic balance condition presents no formal difficulties. Upon reference to equations (1.31) - (1.36), we may observe that the spectral expansion of r contains $P_L^M(\cos\theta)$ factors. Such factors give a cosine term when differentiated with respect to θ cancelling the ($\cos\theta$) on the left of (1.25). The other equation must be examined more carefully, which we will omit, but no difficulties arise.

However, this is not encouraging, since the case M = 0 cannot be worked out, and these coefficients constitute that part of the zonal circulation which is independent of longitude. It is concluded, on the whole, that the geostrophic balance assumption is not suitable for use in VSH modeling.

Further Development of the Advective Term.

The advective term $(\vec{v}\cdot\nabla)\vec{v}$ has been derived before and a typical term has the form

$$\left(f^{M}_{JL}\vec{T}^{M}_{JL}\cdot\nabla\right)f^{M'}_{J'L'}\vec{T}^{M'}_{J'L'} = (4\pi)^{-\frac{1}{2}} f^{M}_{JL}(2J'+1)^{\frac{1}{2}}(-1)^{L'+J'} \tag{2-1}$$

$$\cdot \sum_{L'',J''} W(L'J'L''J'',1J)\langle JMJ'M'|J''(M+M')\rangle(-1)^{\frac{L-L'-L''+1}{2}}$$

$$\cdot \left\{ Z(LJL'-1\,L';1L'')\left(\frac{L'}{2L'+1}\right)^{\frac{1}{2}}\left(\frac{d}{dr}+\frac{L'+1}{r}\right)\right.$$

$$\left. + Z(LJL'+1\,L';1L'')\left(\frac{L'+1}{2L'+1}\right)^{\frac{1}{2}}\left(\frac{d}{dr}-\frac{L'}{r}\right)\right\} f^{M'}_{J'L'}\vec{T}^{M+M'}_{J''L''}$$

When (2-1) is actually computed there appear some simplifications which do not seem to be derivable, at least not with a reasonable amount of labor, from the properties of the various vector coupling coefficient. In order to investigate these simplifications we have employed a different method which we will now describe. The idea is to try to capitalize on the fact that in addition to the $\vec{T}^{M}_{JL}$ representation to which is orthogonal in the function sense one can use the representation in terms of the linear combination

$$\vec{A}^{M}_{L}(\theta,\varphi),\ \vec{B}^{M}_{L}(\theta,\varphi) \text{ and } \vec{C}^{M}_{L}(\theta,\varphi)$$

discussed before, which in addition to being orthogonal in the function sense also an orthogonal in a geometric sense.

It is this last property that we want to investigate further.

We repeat one of the definitions of these functions:

$$A^M_L(\theta,\varphi) \equiv \hat{e}_r\, Y^M_L(\theta,\varphi) \tag{2-2}$$

$$B^M_L(\theta,\varphi) \equiv \frac{r\nabla Y^M_L(\theta,\varphi)}{\gamma_L} = \frac{1}{\gamma_L}\left[\hat{e}_\theta \frac{\partial Y^M_L}{\partial\theta} + \hat{e}_\varphi \frac{1}{\sin\theta}\frac{\partial Y^M_L}{\partial\varphi}\right] \tag{2-3}$$

$$C^M_L(\theta,\varphi) \equiv -\frac{i\,\vec{r}\times\nabla Y^M_L(\theta,\varphi)}{\gamma_L} = -\frac{i}{\gamma_L}\left[\hat{e}_\varphi \frac{\partial Y^M_L}{\partial\theta} - \hat{e}_\theta \frac{1}{\sin\theta}\frac{\partial Y^M_L}{\partial\varphi}\right] \tag{2-4}$$

$$\gamma_L = \sqrt{L(L+1)} \tag{2-5}$$

In spherical coordinators the del operator is:

$$\nabla = \hat{e}_r \frac{\partial}{\partial r} + \frac{\hat{e}_\theta}{r}\frac{\partial}{\partial\theta} + \frac{\hat{e}_\varphi}{r}\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi} \tag{2-6}$$

The A's are radial functions and the B's and C's are tangential to the surface of a sphere.

Let us break up the velocity field into two parts. We will now use the indices 1 and 2 to refer to vertical and tangential components, respectively.

$$\vec{V} = \vec{V}_1 + \vec{V}_2 \tag{2-7}$$

$$\nabla = \nabla_1 + \nabla_2 \tag{2-8}$$

$$\vec{V}_1 = \sum a^M_L(r,t) A^M_L(\theta,\varphi) \tag{2-9}$$

$$V_2 = \sum b^M_L(r,t) B^M_L(\theta,\varphi) + \sum C^M_L(r,t) C^M_L(\theta,\varphi) \tag{2-10}$$

$$\nabla_1 \equiv \hat{e}_r \frac{\partial}{\partial r} \quad ; \quad \nabla_2 \equiv \frac{\hat{e}_\theta}{r} \frac{\partial}{\partial \theta} + \frac{\hat{e}_\varphi}{r} \frac{1}{\sin\theta} \frac{\partial}{\partial \varphi} \tag{2-11}$$

We can then write $(\vec{V}\cdot\nabla)\vec{V}$ as:

$$\left((\vec{V}_1+\vec{V}_2)\cdot(\nabla_1+\nabla_2)\right)(\vec{V}_1+\vec{V}_2) = \vec{V}_1\cdot\nabla_1\vec{V}_1 + (\vec{V}_1\cdot\nabla_1)\vec{V}_2 \tag{2-12}$$

$$+ (\vec{V}_1\cdot\nabla_2)\vec{V}_1 + (\vec{V}_1\cdot\nabla_2)\vec{V}_2 + (\vec{V}_2\cdot\nabla_1)\vec{V}_1 + (\vec{V}_2\cdot\nabla_1)\vec{V}_2$$

$$+ (\vec{V}_2\cdot\nabla_2)\vec{V}_1 + (\vec{V}_2\cdot\nabla_2)\vec{V}_2$$

Let us consider each of these eight terms separately:

$$(\vec{V}_1\cdot\nabla)\vec{V}_1 = \sum_{L,M} a^M_L \hat{e}_r Y^M_L \cdot \hat{e}_r \frac{\partial}{\partial r} \sum_{L'M'} a^{M'}_{L'} Y^{M'}_{L'} \tag{2-13}$$

$$= \sum_{LM}\sum_{L'M'} a^M_L \frac{\partial a^{M'}_{L'}}{\partial r} \hat{e}_r Y^M_L Y^{M'}_{L'}$$

$$= \frac{1}{\sqrt{4\pi}} \sum_{LM}\sum_{L'M'} a^M_L \frac{\partial a^{M'}_{L'}}{\partial r} [L][L'] \sum_{L''} \frac{1}{[L'']} \langle L0L'0|L''0\rangle\langle LML'M'|L''M+M'\rangle A^{M+M'}_{L''}$$

Where

$$[L] \equiv \sqrt{2L+1}$$

We thus find that the advective term of the radial velocity with itself is radial. Further we see that the $1/r$ term in the general expression is in fact zero for the combination of $\vec{T}^{M}_{JL}$ which is vertical, namely

$$A^{M}_{L}(\theta,\varphi) = \beta_L \vec{T}^{M}_{L\,L-1}(\theta,\varphi) - \alpha_L \vec{T}^{M}_{L\,L+1}(\theta,\varphi) \qquad \text{(2-14)}$$

This allows one to show that there are additional relationships between the coupling coefficients which were unknown before. We postpone the investigation of these coefficients. Now consider: $(\vec{V}\cdot\nabla_2)\vec{V}_1$

$$(\vec{V}_1\cdot\nabla_2) = \sum a^{M}_{L}(r,t)\,\hat{e}_r\, Y^{M}_{L}(\theta,\varphi)\cdot\left[\frac{\hat{e}_\theta}{r}\frac{\partial}{\partial\theta} + \hat{e}_\varphi\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\right]$$

but

$$\hat{e}_r \cdot \hat{e}_\theta = \hat{e}_r \cdot \hat{e}_\varphi = 0$$

Thus we have the result

$$(\vec{V}_1 \cdot \nabla_2)\vec{V}_1 = (\vec{V}_1 \cdot \nabla_L)\vec{V}_2 = 0 \qquad (2\text{-}15)$$

Consider $(\vec{V}_1 \cdot \nabla_1)\vec{V}_2$:

$$(\vec{V}_1 \cdot \nabla_1)\vec{V}_2 = \left(\sum \hat{e}_r\, a^M_L(r,t)\, \psi^M_L(\theta,\varphi) \cdot \hat{e}_r \frac{\partial}{\partial r}\right)\vec{V}_2$$

$$= \sum a^M_L(r,t)\, \psi^M_L(\theta,\varphi) \frac{\partial}{\partial r}\Big(\sum b^M_L(r,t)\, \Phi^M_L(\theta,\varphi)$$

$$+ \sum c^M_L(r,t)\, C^M_L(\theta,\varphi)\Big) \qquad (2\text{-}16)$$

We see, then, that $(\vec{V}_1 \cdot \nabla_1)\vec{V}_2$ is tangential and again the $1/r$ terms are missing.

Consider $(\vec{V}_2 \cdot \nabla_1)\vec{V}_1$

$\vec{V}_2$ is tangential so $(\vec{V}_2 \cdot \nabla_1)$ has the form

$$(\hat{e}_\theta f_1(r,\theta,\varphi,t) + \hat{e}_\varphi f_2(r,\theta,\varphi,t)) \cdot \hat{e}_r \frac{\partial}{\partial r} \tag{2-17}$$

$$\hat{e}_\theta \cdot \hat{e}_r = \hat{e}_\varphi \cdot \hat{e}_r = 0 \quad \text{thus} \tag{2-18}$$

$$(\vec{V}_2 \cdot \nabla_1)\vec{V}_1 = (V_2 \cdot \nabla_1)\vec{V}_2 = 0$$

Consider $(\vec{V}_2 \cdot \nabla_2)\vec{V}_1$

$$(V_2 \cdot \nabla_2)\vec{V}_1 = \left(\sum b^M_L(r,t)\, B^M_L(\theta,\varphi) + \sum c^M_L(r,t)\, C^M_L(\theta,\varphi)\right)$$

$$\cdot \frac{1}{r}\left[\hat{e}_\theta \frac{\partial}{\partial\theta} + \frac{\hat{e}_\varphi}{\sin\theta}\frac{\partial}{\partial\varphi}\right]\left(\sum a^{M'}_{L'}(r,t)\,\hat{e}_r\, Y^M_L(\theta,\varphi)\right) \tag{2-20}$$

$$\frac{\partial \hat{e}_r}{\partial\theta} = \frac{2\hat{e}_\theta}{r}$$

Thus this expression has both radial and tangential components but no derivatives.

Fianlly consider $(\vec{V}_2 \cdot \nabla_2)\vec{V}_2$

$$(\vec{V}_2 \cdot \nabla_2)\vec{V}_2 = \sum\left(b^M_L(r,t)\,B^M_L(\theta,\varphi) + \sum C^M_L(r,t)\, C^M_L(\theta,\varphi)\right)$$

$$\cdot \nabla_2\left(b^{M'}_{L'}(r,t)\,B^{M'}_{L'}(\theta,\varphi) + c^{M'}_{L'}(r,t)\, C^{M'}_{L'}(\theta,\varphi)\right) \tag{2-24}$$

This gives radial and tangential fields. Since one can expand the B's, C's in T's and then only have derivatives with respect to r and there are no derivatives with respect r in the formulation above it means that there can be no derivatives at all in the general formulation of this term.

We see then that the only terms in $(\vec{V}\cdot\nabla)\vec{V}$ containing derivatives are

$$(\vec{V}_1\cdot\nabla)\vec{V}_1 \quad \text{and} \quad (\vec{V}_1\cdot\nabla)\vec{V}_2$$

all the other have a factor $1/r$ and if r is taken to be the radius of the earth one sees that there contributions are small.

We will now consider how the information obtained above can be applied to simplyifying (1) and also to the derivation of terms like $(\vec{V}_2\cdot\nabla)\vec{V}_2$

First consider

$$\left(\left(C^{M}_{L}(r,t)\, C^{M}_{L}(\theta,\varphi)\right)\cdot\nabla\right) C^{M}_{L}(r,t)\, C^{M}_{L}(\theta,\varphi)$$

which is the simpliest of the terms involved.

$$C^{M}_{L}(\theta,\varphi) \equiv T^{M}_{LL}(\theta,\varphi)$$

$$\left(C^{M}_{L}(r,t)\, C^{M}_{L}(\theta,\varphi)\cdot\nabla\right) C^{M'}_{L'}(r,t)\, C^{M}_{L}(\theta,\varphi) = \frac{C^{M}_{L}[L']}{\sqrt{4\pi}} \tag{2-25}$$

$$\cdot\, W(L'LL'J'';1L)\langle LML'M'|L''(M+M')\rangle(-1)^{\frac{L'+1}{2}}\Big\{Z(LLL'-1L';1L'')$$

$$\cdot\,\beta_{L'}D(L'+1) + Z(LLL'+1L';1L'')\alpha_{L'}D(-L')\Big\}\, C^{M'}_{L'}(r,t)\, T^{M+M'}_{J''L''}(\theta,\varphi)$$

Since the d/dr terms vanish one finds

$$\beta_{L'} Z(LLL'-1L';1L'') + \alpha_{L'} Z(LLL'+1L';1L'') = 0 \tag{2-26}$$

This result has also been derived from the definitions of the Z-coefficients.

The $1/r$ terms then become

$$\left[\beta_{L'}\, Z(L\, L\, L'-1\, L'_1;\, 1\, L'')\,|L'+1| - \alpha_{L'}\, Z(L\, L\, L'+1\, L'_1;\, 1\, L'')\right]\frac{1}{r} \tag{2-27}$$

multipled by the terms outside of the curly brackets in equation (2-26) The other terms can be derived in a similar manner. We will not display he results here since they are obtained from a straightforward manipulation completely analogous to the derivation of (2-26).

II.3 Kinetic Energy and Angular Momentum in VSH Spectral Forms

The section on angular momentum is taken from a previous report (1972). It is included for completeness.

II.3.1 Kinetic Energy

The kinetic energy of the atmosphere is given by the expression

$$2T = \int \rho \, \bar{V} \cdot \bar{V} \, d\tau \tag{3-1}$$

where T is kinetic energy and $d\tau$ is a volume element. The integration is over all the atmosphere. As usual, we expand $\bar{V}$ in vector spherical harmonics:

$$\bar{V} = \sum_{M,L} a_L^M A_L^M + b_L^M B_L^M + c_L^M C_L^M \tag{3-2}$$

We will assume that ρ is independent of θ and φ, and a spherical earth, then

$$2T = \sum_{\substack{M M' \\ L L'}} \int_{r_0}^{r} \int_0^{2\pi} \int_0^{\pi} \rho(r) \left(a_L^M A_L^M + b_L^M B_L^M + c_L^M C_L^M \right) \cdot \left(a_{L'}^{M'} A_{L'}^{M'} + b_{L'}^{M'} B_{L'}^{M'} + c_{L'}^{M'} C_{L'}^{M'} \right) dr \, r^2 \sin\theta \, d\theta \tag{3-3}$$

The expressions for the dot products of A_L^M, B_L^M and C_L^M can be represented as follows:

$$A_L^M \cdot A_{L'}^{M'} = \sum_{L''} C_{AA} Y_{L''}^{M+M'} \tag{3-4}$$

$$A_L^M \cdot B_{L'}^{M'} = A_L^M \cdot C_{L'}^{M'} = 0 \tag{3-5}$$

$$B_L^M \cdot C_L^M = - C_L^M \cdot C_{L'}^{M'} = \sum C_{BB} Y_{L''}^{M+M'} \tag{3-6}$$

$$B_L^M \cdot C_{L'}^{M'} = - C_L^M \cdot B_{L'}^{M'} = \sum C_{BC} Y_{L''}^{M+M'} \tag{3-7}$$

where the C_{BB}, etc. depend on L, L', L''.

In addition, the complex conjugate relations may be shown to be

$$A_L^{-M} = (-)^M A_L^{M*} \tag{3-8}$$

$$B_L^{-M} = (-)^M B_L^{M*} \tag{3-9}$$

$$C_L^{-M} = (-)^{M+1} C_L^{M*} \tag{3-10}$$

We also will require the fact that when $L = L'$ and $M' = -M$, the coefficients C_{AA}, C_{BB}, etc., reduce to

$$C_{AA} = C_{BB} = (-)^M / \sqrt{4\pi} \tag{3-11}$$

Upon inserting the expressions (3-5) through (3-7) in equation (3-3) we obtain:

$$2T = \sum_{\substack{ML \\ M'L' \\ L''}} \int_{r_0}^{\infty}\int_0^{2\pi}\int_0^{\pi} \rho(r)\left[a_L^M a_{L'}^{M'} C_{AA} + b_L^M b_{L'}^{M'} C_{BB} - C_{BB}\, c_L^M c_L^{M'} \right] Y_{L''}^{M+M'}\, dr\, d\varphi \sin\theta\, d\theta \tag{3-12}$$

The remaining cross terms will cancel due to (3-7). We may now express any term in (3-12) as

$$\int_{r_0}^{\infty} f(r)\, dr \int_0^{2\pi}\int_0^{\pi} Y_{L''}^{M+M'}\, d\varphi \sin\theta\, d\theta \tag{3-13}$$

and note that this integral is zero unless $M' = -M$ and $L'' = 0$. Since $|L - L'| \leq L'' \leq L + L'$ we require $L = L'$ and $M' = -M$ and all other terms vanish. Equation (3-12) is then

$$2T = \sum_{ML} \sqrt{4\pi} \int_{r_0}^{\infty}\int_0^{\pi} Y_0^0\, d\varphi \sin\theta\, d\theta \tag{3-14}$$

where $$\sqrt{4\pi} = \int_0^{2\pi}\int_0^{\pi} Y_0^0 \, d\varphi \sin\vartheta \, d\vartheta$$

When we use (3-8) through (3-11) this reduces simply to

$$2T = \sum_{L,M\geq 0} \int_{r_0}^{r} \rho(r)\left[a_L^M a_L^{M*} + b_L^M b_L^{M*} + c_L^M c_L^{M*}\right] dr \qquad (3\text{-}15)$$

Thus, the kinetic energy reduces to a single integral over r. All terms carry energy. This relation is analogous to other spectral expansions, where the energy is proportional to the sums of the squares of the coefficients.

II.3.2 Angular Momentum

This section is condensed from the report for 1972.

The angular momentum is defined by

$$\bar{L} = \int \bar{r} \times \rho \bar{v} \, d\tau \qquad (3\text{-}13)$$

as before, we assume a spherical earth with density spherically symmetric. Expanding $\bar{v}$ using (3-2) we have

$$\bar{L} = \int r\hat{e}_r \times \rho(r) \sum_{M,L} \left[a_L^M A_L^M + b_L^M B_L^M + c_L^M C_L^M\right] d\tau \qquad (3\text{-}14)$$

Evaluating the integrals yields

$$= \sum_{M,L} \int r\hat{e}_r \times \rho a_L^M A_L^M + \int r\hat{e}_r \times \rho b_L^M B_L^M + \int r\hat{e}_r \times \rho c_L^M C_L^M \qquad (3\text{-}15)$$

using definitions of A, B, C gives

$$= \sum_{M,L} \int r\rho a_L^M \, \hat{e}_r \times \hat{e}_r Y_L^M + \int r^2 \rho \frac{b_L^M}{\gamma_L} \bar{r} \times \nabla Y_L^M + \int r^2 \rho c_L^M (e_r \times \bar{r} \times \nabla Y_L^M) \qquad (3\text{-}16)$$

$$= \sum_{M,L} 0 + \int f_1(r) dr \int c_L^M d\Omega + \int f_2(r) dr \int b_L^M d\Omega \qquad (3\text{-}17)$$

$$= \quad \text{terms in } C_L^M \text{ with } Y_0^0 + \text{terms in } B_l^M \text{ with } Y_0^0$$

$$= \quad B_1^{-1},\ B_1^0,\ B_1^1$$

Thus, the only terms carrying angular momentum are

$$C_1^{-1},\quad C_1^0,\quad C_1^{+1}$$

Other terms may carry angular momentum locally, but must cancel in the integrals.

II.4 Analysis of Sound Waves in VSH

This section applies Vector Spherical Harmonics to sound waves. No startling results are discovered but some light is shed on how sound waves behave in terms of VSH.

II.4.1 Derivation of the Sound Wave Equation from the Hydrodynamic Equations

Following Landau and Lifschitz[1], let

$$p = p_0 + p' \;,\; \rho = \rho_0 + \rho' \;;\; \rho_0 >> \rho' \;,\; p_0 >> p' \tag{4-1}$$

The continuity equation is $\frac{\partial \rho}{\partial t} = \nabla (\rho \bar{v})$ (4-2)

or
$$\frac{\partial}{\partial t}(\rho_0 + \rho') = \frac{\partial \rho'}{\partial t} = \nabla \cdot [(\rho_0 + \rho')\bar{v}] \approx \rho_0 \nabla \cdot \bar{v} \tag{4-3}$$

The equation of motion used is

$$\frac{\partial \bar{v}}{\partial t} + (\bar{v} \cdot \nabla)\bar{v} = -\frac{\nabla p}{\rho} \tag{4-4}$$

or approximately, neglecting $(\bar{v} \cdot \nabla)\bar{v}$ and using (4-1)

$$\frac{\partial \bar{v}}{\partial t} \approx \frac{-\nabla(p_0 + p')}{\rho' + \rho_0} \simeq -\frac{\nabla p'}{\rho_0} \tag{4-5}$$

We next eliminate ρ' by using the assumption that the process is adiabatic. Then

$$dp = \left(\frac{\partial p}{\partial \rho_0}\right)_s d\rho \tag{4-6}$$

Where s denotes the entropy. Thus

$$p' = \left(\frac{\partial p}{\partial \rho_0}\right)_s \rho' \quad \text{or} \tag{4-7}$$

$$p' = \kappa \rho' \quad , \quad \kappa \equiv \left(\frac{\partial p}{\partial \rho_0}\right)_s \tag{4-8}$$

The equations for sound are then:

$$\frac{\partial p'}{\partial t} = \kappa \rho_0 \, \nabla \cdot \bar{V} \tag{4-9}$$

$$\frac{\partial \bar{V}}{\partial t} = -\frac{1}{\rho_0} \nabla p' \tag{4-10}$$

II.4.2 Analysis in Terms of Vector Spherical Harmonics

First we show that the velocity $\bar{V}$ has no C_L^M terms.

Let

$$\bar{V} = \nabla \varphi + \nabla \times \bar{A} \tag{4-11}$$

$$V_1 \equiv \nabla \times \bar{A} \tag{4-12}$$

Obviously the part of $\bar{V}$ due to $\nabla\varphi$ has no C_L^M's. For the V_1-part, let

$$\bar{V}_1 = \sum a_L^M A_L^M + b_L^M B_L^M + c_L^M C_L^M \tag{4-13}$$

$$\bar{A} = \sum \alpha_L^M A_L^M + \beta_L^M B_L^M + \gamma_L^M C_L^M \tag{4-14}$$

So (10) is now

$$\frac{\partial}{\partial t}\left(\nabla \varphi + \nabla \times \bar{A} \right) = \frac{-1}{\rho_0} \nabla p \tag{4-15}$$

or

$$\frac{\partial}{\partial t} \nabla \varphi + \frac{\partial}{\partial t} (\nabla \times \bar{A}) = \frac{1}{\rho_0} \nabla p \tag{4-16}$$

Since $\nabla\varphi$ and $\nabla p/\rho_0$ have no C_L^M terms, it follows that

$$\left[\frac{\partial}{\partial t}(\nabla \times A)\right]_C = 0 \qquad (4\text{-}17)$$

Now

$$\nabla \times \bar{A} = \left(-i\frac{\gamma_L}{r}\alpha_L^M + i b_1 \beta_L^M\right) C_L^M + \frac{i\gamma_L}{r}\gamma_L^M A_L^M + i b_1 \gamma_L^M B_L^M$$

Therefore

$$\frac{\partial}{\partial t}\left(-\frac{i\gamma_L}{r}\alpha_L^M + i b_1 \beta_L^M\right) C_L^M = 0 \qquad (4\text{-}18)$$

But since $\qquad \bar{V}_c = (\nabla \times A)_c$

$$C_L^M = \left(\frac{i\gamma_L}{r}\alpha_L^M + i b_1 \beta_L^M\right) \qquad (4\text{-}19)$$

Thus (19) and (18) imply

$$\frac{\partial c_L^M}{\partial t} = 0 \qquad (4\text{-}20)$$

Therefore, the C_L^M terms in $\bar{V}$ do not support wave motion-in equations (4-9) and (4-10).

II.4.3 General Solution of the Equations in Spherical Coordinates. Equations (4-9) and (4-10) generally describe <u>wave</u> <u>motion</u>. To see this, let

$$\bar{V} = \bar{V}_1 + \bar{V}_2 \qquad (4\text{-}21) \quad \text{where} \quad \nabla\cdot\bar{V}_1 = 0 \qquad (4\text{-}22)$$

$$\nabla \times \bar{V}_2 = 0 \qquad (4\text{-}23)$$

Then (10) is

$$\frac{\partial \bar{v}_1}{\partial t} + \frac{\partial \bar{v}_2}{\partial t} = -\frac{1}{\rho_0} \nabla p \tag{4-24}$$

Where we write p for convenience (we really mean p'), and (4-9) is

$$\frac{\partial p}{\partial t} = -k \rho_0 \nabla \cdot \bar{v}_2 \tag{4-25}$$

Taking the divergence of (4-24) we have

$$\frac{\partial}{\partial t} (\nabla \cdot \bar{v}_2) = -\frac{1}{\rho_0} \nabla^2 p \tag{4-26}$$

From (25), $\nabla \cdot \bar{v}_2 = \frac{1}{k\rho_0} \frac{\partial p}{\partial t}$, so substituting into (4-26) gives

$$\frac{\partial}{\partial t} \left(\frac{1}{k\rho_0} \right) \frac{\partial p}{\partial t} = - \frac{\nabla^2 p}{\rho_0}$$ or

$$\frac{\partial^2 p}{\partial t^2} = k \nabla^2 p \tag{4-27}$$

This is the scalar wave equation for p.

Now we take the gradient of (4-26):

$$\frac{\partial}{\partial t} (\nabla p) = k\rho_0 \nabla (\nabla \cdot \bar{v}_2) \tag{4-28}$$

Now, since

$$\nabla^2 \bar{v}_2 = \nabla (\nabla \cdot v_2) - \nabla \times \nabla \times \bar{v}_2 \tag{4-29}$$

and $\nabla \times \bar{v}_2 = 0$

and since $\nabla p = \rho_0 (\partial_t \bar{v}_1 + \partial_t \bar{v}_2)$ from (4-24), (4-28) becomes

$$\frac{\partial}{\partial t} \left(\frac{\partial \bar{v}_1}{\partial t} + \frac{\partial \bar{v}_2}{\partial t} \right) = k \nabla^2 \bar{v}_2 \tag{4-30}$$

Taking the curl of (24) we find

$$\frac{\partial}{\partial t}(\nabla \times V_1) = 0 \tag{4-31}$$

Thus $\nabla \times \bar{V}_1 =$ const. in time

or $\bar{V}_1$ = const. in time since curl doesn't affect time behavior

Thus (30) is now

$$\frac{\partial^2 \bar{V}_2}{\partial t^2} = k \nabla^2 \bar{V}_2 \tag{4-32}$$

and $\bar{V}_1$, the "incompressible" part of $\bar{V}$, does not support sound waves. We now solve (4-27) in spherical coordinates. Let

$$p = \sum \varphi_L^M(r,t)\, Y_L^M(\theta,\varphi) \tag{4-33}$$

(4-33) substituted in (4-27) gives an equation for the coefficients :

$$\frac{\partial^2}{\partial t^2}\varphi_L^M = k\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{L(L+1)}{r^2}\right)\varphi_L^M \tag{4-34}$$

Let

$$\varphi_L^M(r,t) = \varphi_{tL}^M(t)\, \varphi_{rL}^M(r) \tag{4-35}$$

then

$$\text{(4-34)} = \frac{\partial^2 \varphi_{tL}^M}{\partial t^2}\varphi_{rL}^M(r) = k\left[\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{L(L+1)}{r^2}\right)\right]\varphi_{rL}^M\, \varphi_{tL}^M \tag{4-36}$$

or

$$C = \frac{\ddot{\varphi}_{tL}^M(t)}{k\varphi_{tL}^M(t)} = \frac{\left(\partial_r^2 + 2/r\,\partial_r - L(L+1)/r^2\right)\varphi_{rL}^M(r)}{\varphi_{rL}^M(r)} \tag{4-37}$$

Thus

$$\ddot{\varphi}_{tL}^M(t) + Ck\,\varphi_{tL}^M(t) = 0 \tag{4-38}$$

$$\left[\partial_r^2 + \frac{2}{r}\partial_r - \frac{L(L+1)}{r^2} + C \right] \varphi^M_{rL}(r) = 0 \tag{4-39}$$

The solution to (4-38) is

$$\varphi^M_{tL}(t) = \Phi^M_{t1L}\, e^{\sqrt{-\kappa C}\, t} + \Phi^M_{t2L}\, e^{-\sqrt{-\kappa C}\, t} \tag{4-40}$$

and the solution to (4-39) is

$$\varphi^M_{rL}(r) = \Phi^M_{rL1}\, j_L(\sqrt{C}\, r) + \Phi^M_{rL2}\, y_L(\sqrt{C}\, r) \tag{4-41}$$

j_L, y_L are Spherical Bessel functions of first and second kind respectively.

We can also solve (4-32). It is more convenient to use the T^M_{JL} rather than A^M_L, B^M_L, C^M_L since the former are eigenfunctions of ∇^2.

When we let

$$\bar{v} = \sum_{MJL} v^M_{JL}\, T^M_{JL} \tag{4-42}$$

we obtain

$$\sum_{MJL} \Big\{ \partial_t^2 v^M_{LL-1} T^M_{LL-1} + \partial_t^2 v^M_{LL} T^M_{LL} + \partial_t^2 v^M_{LL+1} T^M_{LL+1} = \kappa D_{L+1} D_{1-L}\, v^M_{LL-1} T^M_{LL-1}$$

$$- (\alpha_L^2 D_{1-L} D_{L+1} + \beta_L^2 D_{L+2} D_{-L})\, v^M_{LL} T^M_{LL} + D_{-L} D_{L+2}\, v^M_{LL+1} T^M_{LL+1} \Big\} \tag{4-43}$$

The equations for the coefficients are:

$$\frac{\partial^2 v^M_{LL-1}}{\partial t^2} = \kappa D_{L+1} D_{1-L}\, v^M_{LL-1} \tag{4-44}$$

$$\frac{\partial^2 v^M_{LL}}{\partial t^2} = -k\left(\alpha_L^2 D_{1-L} D_{L+1} + \beta_L^2 D_{L+2} D_{-L}\right) v^M_{LL}$$

$$\frac{\partial^2 v^M_{LL+1}}{\partial t^2} = k D_{-L} D_{L+2} V^M_{LL+1} \tag{4-45}$$

We let
$$V^M_{JL} = V^M_{tJL}(t)\, V^M_{rJL}(r) \tag{4-46}$$

and obtain the following separated equations. The constant of separation is C_{L-1}, C_L, C_{L+1} respectively; or C_J in general.

We know in fact that the $C^M_L = T^M_{LL}$ terms will not be present from equation (4-20) (or will be constant).

$$\ddot{V}^M_{tJL}(t) + kC_J V^M_{tJL}(t) = 0 \tag{4-47}$$

and for the radial parts:

$$\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{L(-L+1)}{r^2} + C_{L-1}\right) v^M_{rLL-1} = 0 \tag{4-48}$$

$$\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{L(L+1)}{r^2} + C_L\right) v^M_{rLL} = 0 \tag{4-49}$$

$$\left(\frac{\partial}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{(L+1)(L+2)}{r^2} + C_{L+1}\right) v^M_{rLL+1} = 0 \tag{4-50}$$

and therefore:

$$V^M_{tJL}(t) = V^M_{tJL1}\, e^{\sqrt{-kC_J}\, t} + V^M_{tJL2}\, e^{-\sqrt{-kC_J}\, t} \tag{4-51}$$

$$v^M_{r\,L\,L-1} = V^M_{1rLL-1}\, j_{L-1}(\sqrt{C_{L-1}}\, r) + V^M_{2rLL-1}\, y_{L-1}(\sqrt{C_{L-1}}\, r) \tag{4-52}$$

$$v^M_{r\,LL} = V^M_{1rLL}\, j_L(\sqrt{C_L}\, r) + V^M_{2rLL}\, y_L(\sqrt{C_L}\, r) \tag{4-53}$$

$$v^M_{r\,LL+1} = V^M_{1rLL+1}\, j_{L+1}(\sqrt{C_{L+1}}\, r) + V^M_{2rLL+1}\, y_{L+1}(\sqrt{C_{L+1}}\, r) \tag{4-54}$$

Where C_{L-1}, C_L, C_{L+1} must be determined from boundary conditions; and are spherical Bessel functions of the first and second kind. Since in fact we know that T^M_{LL}'s are not present we can dispose of (4-53) in considering wave motion.

II.4.4 The Velocity Potential

Since the T^M_{LL}'s are not in fact present, we can write $\bar{v}$ as the gradient of a scalar velocity potential, paralleling conventional developments. (Note that it isn't necessary.) When we do this we have

$$\bar{v} = \nabla\psi \tag{4-55}$$

$$\frac{\partial p}{\partial t} = -k\rho_0\, \nabla\cdot\bar{v} \tag{4-9}$$

$$\frac{\partial \bar{v}}{\partial t} = \frac{-1}{\rho_0}\nabla p \tag{4-10}$$

Then

$$\frac{\partial p}{\partial t} = -k\rho_0\,(\nabla\cdot(\nabla\psi)) = -k\rho_0\,\nabla^2\psi \tag{4-56}$$

$$\frac{\partial(\nabla\psi)}{\partial t} = \nabla\left(\frac{\partial\psi}{\partial t}\right) = -\frac{1}{\rho_0}\nabla p \tag{4-57}$$

So that aside from a proportionality factor,

$$p = -\rho_0\frac{\partial\psi}{\partial t} \tag{4-58}$$

cf. Ref. (1), eq. 63.6. Using (4-58) and (4-56) we get

$$\frac{\partial p}{\partial t} = \rho_0\frac{\partial^2\psi}{\partial t^2} = k\rho_0\nabla^2\psi \quad \text{or}$$

$$\frac{\partial^2 \psi}{\partial t^2} - k \nabla^2 \psi = 0 \qquad (4\text{-}59)$$

Which is identical to the pressure equation (4-27) so its solution is given by (4-40) and (4-41), with different constants.

II.4.5 Boundary Conditions

Note that in (52) and (54) we have the constants C_{l-1} & C_{l+1}. We have to infer those constants from boundary conditions. If the boundary condition is

$$\bar{V} \cdot \hat{e}_r = 0 \quad , r = r_0$$

that implies

$$a_l^M(r_0) = \beta_l \, v_{ll-1}^M(r_0, t) - \alpha_l \, v_{ll+1}^M(r_0, t) = 0 \qquad (4\text{-}60)$$

or

$$\beta_l V_{1rll-1}^M J_{l-1}(\sqrt{c_{l-1}}\, r_0) - \alpha_l V_{1rll+1}^M J_{l+1}(\sqrt{c_{l+1}}\, r_0) +$$

$$\beta_l V_{2rll-1}^M y_{l-1}(\sqrt{c_{l-1}}\, r_0) - \alpha_l V_{2rll+1}^M y_{l+1}(\sqrt{c_{l+1}}\, r_0) = 0 \qquad (4\text{-}61)$$

This yields one equation relating C_{l-1} and C_{l+1}. With the boundary condition chosen, C_{l+1}, C_{l-1} cannot be determined. This conforms to the fact that the wavelength (or frequency) of sound waves on the surface of a sphere have no characteristic frequencies. If we were to impose an additional condition, e.g., $\bar{V} \cdot \hat{e}_r = 0$ when $r = r_1$, we would have an additional equation, similar to (4-61), relating C_{l+1}, C_{l-1}; which would determine the characteristic frequencies.

REFERENCES FOR SECTION II

(1) Landau and Lifschitz, Fluid Mechanics. Addison-Wesley, N.Y. (1959)

III. Primitive Equations Meteorological Modeling

This section discusses efforts to develop a numerical weather prediction model based on a dry-air version of the Mintz-Arakawa Model [1] utilizing the method of Vector Spherical Harmonics for the solution of equations.

A previous report[2] described a quasi-analytic solution of the equations. For the current effort, the level of implementation is confined to the framework of the General Program System described in the same report. The modeling activity serves as a test for the general system.

In analyzing the Mintz-Arakawa model, it appears that it is distinguished from other models by (1) The forcing fields assumed by the model, i.e. humidity input, radiative heating, and friction loss terms; (2) The horizontal differencing technique; and (3) the method of handling convective adjustment. Of these aspects (2) is not applicable to Vector Spherical Harmonics and (3) does not apply in view of the dry-air assumption which also removes humidity inputs. Since so much of Mintz-Arakawa proper does not apply to this effort, it was deemed constructive - especially given the fact that a novel solution technique was to be used - to begin with the basic primitive equations in spherical coordinates and adapt them for VSH solution. As a result of several trials, a model, or set of equations, almost identical to Mintz-Arakawa was evolved. The following sections describe this evolution in some detail. As an interesting result, the primitive equations have been rigorously converted to spherical coordinates, using successively the radius, the pressure, and the normalized pressure (σ) as a vertical coordinate. In the process of conversion, a number of terms are obtained which are usually neglected in the literature. In the final version of the model described herein, these additional terms have dropped to maintain consistency with the Mintz-Arakawa model.

Section III.1 states the assumptions used to derive the equations; Section III.2 presents the derivation using r as a vertical coordinate without the hydrostatic balance condition. Section III.3 discusses boundary conditions, Section III.4 the limitations of the model. In Section III.5, the hydrostatic balance condition is imposed and the changes to the model are described. Section III.6 covers the change to pressure and sigma coordinates. Section III.7 presents a functional flow of the implementation.

III.1 Assumptions

The assumptions used to derive all equations in the following sections are:

- Dry Air. No humidity terms are considered in any equation.
- No orography. (in radial coordinates)
- Spherical earth.
- The earth is assumed to be "all oceans". That is, the surface temperature will be taken as a constant function of θ, φ.

In deriving the first set of equations - which form the basis for much of the rest of the work - the Hydrostatic Balance approximation will not be used. Also, in the initial equations, the earth's gravitational field is arbitrary.

The terminology in the following section is given below:

$\bar{V}$	three-diminsional velocity vector field	(meter sec^{-1})
ρ	density field	(gm $meter^{-3}$)
$\bar{\Omega} = \Omega\hat{k}$	earth's angular velocity vector	(sec^{-1})
p	pressure field	(gram $meter^{-1} sec^{-2}$)
T	temperature field	(°K)
R	universal gas constant	($meter^{+2} sec^{-2}$(K$)^{-1}$)
k	Ratio of Cp/Cv	(specific heats)
φ_G	gravitational constant GM_e, where M_e = mass of earth; G = universal constant (gravitational)	($m^3 sec^{-2}$)
$\vec{F}$	frictional loss term vector field	(m sec^{-2})
$\dot{Q}$	heat input function/temperature	(sec^{-1})
$\dot{H}$	heat input	(ly/sec)
ϕ	earth gravitational potential	

Unless otherwise stated, all fields are functions of r, θ, φ and t, and all vector fields are three dimensional.

III.2 Derivation of Equations

The basic set of equations to be used are:

The equation of motion,

$$\frac{\partial \bar{V}}{\partial t} = -(\bar{V}\cdot\nabla)\bar{V} - \frac{1}{\rho}\nabla p + \nabla\phi - 2\bar{\Omega}\times\bar{V} + \bar{F} \qquad (2\text{-}1)$$

The equation of continuity,

$$\frac{\partial \rho}{\partial t} = -\nabla\cdot(\rho\bar{V}) \qquad (2\text{-}2)$$

The ideal gas law,

$$p = \rho R T \qquad (2\text{-}3)$$

and the thermodynamic equation,

$$\frac{1}{T}\frac{dT}{dt} - \frac{k}{p}\frac{dp}{dt} = \dot{q}(r,\theta,\phi,t) = \frac{\dot{H}}{C_p T} \qquad (2\text{-}4)$$

Where ϕ is the earth's gravitational field. If the earth is spherical then

$$\phi = \frac{\phi_G}{r} + \frac{1}{2}|\bar{\Omega}\times\bar{r}|^2 \qquad (2\text{-}5)$$

$$\nabla\phi = -\frac{\phi_G\hat{e}_r}{r^2} - \bar{\Omega}\times\bar{\Omega}\times\bar{r} \qquad (2\text{-}6)$$

ϕ_G is a constant; $\bar{\Omega}\times\bar{\Omega}\times\bar{r}$ represents the centrifugal acceleration due to the rotation of the earth.

The above set of equations is not suitable for solution by means of the General Program System; for which it is preferable to have an equation for the time derivative of each variable (vector or scalar). One way to obtain such a system is to eliminate ρ from the equations by means of the gas law (2-3). Using this equation in 2-1 produces:

$$\frac{\partial \bar{V}}{\partial t} = -(\bar{V}\cdot\nabla)\bar{V} - \frac{RT}{\rho}\nabla p - 2\Omega\times\bar{V} + \nabla\phi + \bar{F} \tag{2-7}$$

Equation (2.2) may also be written (see 2-14) as

$$\frac{d\rho}{dt} = -\rho\,\nabla\cdot\bar{V} \tag{2-8}$$

And using the gas law to eliminate dp/dt in (2-4) results in

$$\frac{1}{T}\frac{dT}{dt} - \frac{\kappa\rho K}{p}\frac{dT}{dt} - \frac{\kappa RT}{p}\frac{d\rho}{dt} = \dot{Q} \tag{2-9}$$

Substituting (2-8) into (2-9) and collecting terms gives

$$\frac{dT}{dt} = \frac{T\dot{Q}}{(1-\kappa)} - \frac{\kappa T}{1-\kappa}\nabla\cdot\bar{V} = \frac{1}{1-\kappa}\left(\dot{Q} - \kappa\nabla\cdot\bar{V}\right) \tag{2-10}$$

If we now use the gas law to eliminate dT/dt in (2-4) we obtain

$$\frac{1}{\rho RT}\frac{dp}{dt} + \frac{p}{\rho^2 RT}\frac{d\rho}{dt} - \frac{\kappa}{p}\frac{dp}{dt} = \dot{Q} \tag{2-11}$$

again using (2-8) in (2-11) and collecting terms in dp/dt, we find

$$\frac{1}{p}\frac{dp}{dt} = \frac{1}{1-\kappa}\left(\dot{Q} - \nabla\cdot\bar{V}\right) \tag{2-12}$$

Now, in spectral methods, it is advantageous to avoid divisions by a variable whenever possible. Such a division occurs in the equation of motion where ∇p is divided by p. To avoid this, introduce the variable ψ , defined by

$$\psi \equiv \ln p \tag{2-13}$$

Thus $\nabla p/p = \nabla\psi$ and $1/p \; dp/dt = d\psi/dt$
Finally we use the fact that if f is a scalar field,

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \bar{V}\cdot\nabla f \tag{2-14}$$

The complete set of equations is then;

$$\frac{\partial \bar{V}}{\partial t} = -(\bar{V}\cdot\nabla)\,\bar{V} - KT\nabla\psi - 2\bar{\Omega}\times\bar{V} + \nabla\phi + \bar{F} \tag{2-15}$$

$$\frac{\partial \psi}{\partial t} = -\bar{V}\cdot\nabla\psi + \frac{1}{1-K}\left(\dot{\varphi} - \nabla\cdot\bar{V}\right) \tag{2-16}$$

$$\frac{\partial T}{\partial t} = -\bar{V}\cdot\nabla T + \frac{T}{1-K}\left(\dot{Q} - K\nabla\cdot\bar{V}\right) \tag{2-17}$$

This is a set of 1 vector and 2 scalar equations in vector unknowns $\bar{V}$ and scalar unknowns ψ and T. These equations are suitable for solution using the General Program System. It is interesting to note that (2-16) is linear except for the term (presumably small) $\bar{V}\cdot\nabla\psi$, and that (2-17) could be made linear (to the same approximation) by defining a variable equal to $\ln T$. This change, however, would complicate the equation

of motion. To complete the specification of the problem, we must state the boundary conditions, the form of the heating term $\dot{Q}$ or $\dot{H}$, and the form of the friction loss term.

To implement a model, we must in addition specify the set of radial functions to be used (see Section V), the number of layers to be used, and the layer spacing.

III.3 Boundary Conditions

The boundary conditions to be specified are:

1) The normal velocity component - in the radial direction for a spherical earth - is zero at the earth's surface, or

$$(\bar{V} \cdot \hat{e}_r)_{r=r_0} = 0 \tag{3-1}$$

2) The surface temperature is treated as constant,

$$T(r_0, \varphi, \theta) = \bar{T}_s(\theta, \varphi) \tag{3-2}$$

Where $\bar{T}_s(\theta, \varphi)$ is an input to the program.

3) No boundary conditions on ψ are specified.

The fact that a frictional term is present really requires that the velocity on the earth's surface be zero. In reality, this leads to boundary layer considerations which are best avoided for the moment.

In the Mintz-Arakawa model, the equivalent of the radial velocity ($\dot{\sigma}$) is set to zero at the top of the atmosphere. For equations (2-15) - (2-17) this equivalent step was not performed.

III.4 Model Limitations

Equations (2-15) through (2-17) constitute a complete set of equations, without any balance conditions of any sort. Thus they are subject to wave solutions, that is, the same sort of computational instabilities that give rise to the well-known Courant-Friederichs-Lewy (CFL) condition for the numerical integration of hyperbolic partial differential equations. We may identify the part of the equations that cause trouble by considering only the equations as follows:

$$\frac{\partial \bar{v}}{\partial t} = -\kappa \bar{T}_0 \nabla \psi \tag{4-1}$$

$$\frac{\partial \psi}{\partial t} = \frac{-1}{1-\kappa} \nabla \cdot \bar{v} \tag{4-2}$$

That is, the nonlinear terms have been dropped, the temperature has been held constant, and the coriolis force and all forcing terms have been neglected. In this simplified model, taking the gradient of (4-2) gives

$$\nabla\left(\frac{\partial \psi}{\partial t}\right) = \frac{\partial}{\partial t}(\nabla \psi) = \frac{-1}{1-\kappa} \nabla(\nabla \cdot \bar{v}) = -\frac{1}{1-\kappa} \nabla^2 \bar{v} \tag{4-3}$$

taking $\partial / \partial t$ of (4-1) gives

$$\frac{\partial^2 \bar{v}}{\partial t^2} = -\kappa \bar{T}_0 \frac{\partial}{\partial t}(\nabla \psi) \tag{4-4}$$

combining (4-3) and (4-4) gives

$$\frac{\partial^2 \bar{v}}{\partial t^2} = \frac{\kappa \bar{T}_0}{1-\kappa} \nabla^2 \bar{v} \tag{4-5}$$

This is the second-order wave equation identical to that studied in Section II.4 when dealing with sound waves. In fact, the propagation speed of the wave is given by

$$c = \sqrt{\frac{R T_0}{1-\kappa}} \tag{4-6}$$

Which is approximately the propagation speed of sound waves. Thus the solution contains sound waves.

We can also expect, from the similarity of equations (2-16) and (2-17) that a similar result holds for waves generated by the equation for $\bar{V}$ and the equation for T. The fact that the equation contains a T multiplier makes it more difficult to analyze.

With respect to solution by spectral methods, the problem is reduced to solving (numerically) a set of ordinary differential equations in time. Once the radial functions are chosen, the partial differential equations in the coefficients become ordinary. For each coefficient there are K ordinary differential equations, where K is the number of radial functions chosen. The numerical stability of a solution procedure for such a set of equations may be analyzed by linearizing the system (as in (4-1) and (4-2) and examining the eigenvalues of the resulting coefficient matrix. The stability of the technique is controlled by the largest eigenvalue in the system. This is equivalent to the CFL condition.

The sound wave coupling between these equations is indicated by diagrammatic representation below:

(TIME INTEGRATION)

$$\frac{\partial \bar{V}}{\partial t} = -(\bar{V}\cdot\nabla)\bar{V} - RT\nabla\boxed{\psi} + g - 2\bar{\Omega}\times\bar{V} + \bar{F}$$

(TIME INTEGRATION)

$$\frac{\partial \boxed{\psi}}{\partial t} = -\bar{V}\cdot\nabla\psi + \frac{1}{1-\kappa}\left(\dot{q} - \boxed{\bar{\nabla}\cdot\bar{V}}\right)$$

with a similiar coupling for the T equation. This type of coupling always leads to oscillation, since it in essence corresponds to a simple harmonic oscillator (or a set thereof).

In order to break this coupling, it is customary in meteorology to resort to the hydrostatic balance approximation. The problem, of course, with any sort of constraint upon the equations is that information (possibly meaningful solutions) will be lost. It should be noted that other possibilities exist. For example, in the second equation above one could neglect the term $\nabla \cdot \bar{v}$. This is not equivalent to setting $\nabla \cdot \bar{v} = 0$ (incompressible fluid): but rather in assuming that the term is negligible in the equation above. In terms of vector spherical harmonics, this means that

$$\nabla \cdot \bar{v} = \sum \left[\left(\frac{\partial}{\partial r} + \frac{2}{r} \right) a_L^M + \frac{\gamma_L}{r} b_L^M \right] \approx 0$$

Since a_L^M is very small $[O(10^{-2} \text{ M SEC}^{-1})]$ and r very large, $O(10^6 \text{ M})$, then this implies that the assumption is that $\frac{\partial a_L^M}{\partial r} + b_L^M/r \approx \frac{\partial a_L^M}{\partial r} \approx 0$ because b_L^M are $O(1 \text{ M SEC}^{-1})$, that is that the vertical derivative of the vertical velocity is small. This assumption must be tested experimentally; but the above discussion makes it appear at least plausible.

Other means of avoiding the coupling above are under investigation. It is perhaps possible to devise a suitable integration scheme or to find better approximations.

III.5 The Hydrostatic Balance Condition

The numerical instability of the complete set of equations (2-15) through (2-17) have led meteorologists to impose the Hydrostatic Balance Condition, which in cartesian coordinates is

$$\frac{\partial p}{\partial z} = -\rho g \tag{5-1}$$

or in spherical coordinates,

$$\frac{\partial p}{\partial r} = -\rho g \tag{5-2}$$

In the equivalent set of equations (2-15) - (2-17) this is also equivalent to the equation

$$RT \frac{\partial \psi}{\partial r} = (\nabla \phi)_r \approx -g \tag{5-3}$$

Where the subscript $_r$ denotes the radial part (along $\hat{e}_r$) of $\nabla\phi$, which is usually approximated by the constant acceleration of gravity g.

III.5.1 Effects

The first effect of the hydrostatic balance condition is to break the coupling between equations (2-15) and (2-16). This is because the velocity field is broken up into vertical ($\bar{v}_1$) and horizontal ($\bar{v}_2$) parts and the vertical part of equation (2-15) is "replaced" by the hydrostatic balance condition. This is a rather glib statement, in as much as (5-2) is not an equation of motion.

It is not immediately clear how (5-2) may replace the vertical motion part of (5-3). One possibility is in fact the set of equations

$$\frac{\partial \bar{v}_2}{\partial t} = -[(\bar{v}\cdot\nabla)\bar{v}]_2 - (2\bar{\Omega}\times\bar{v})_2 - (RT\nabla\psi)_2 + \bar{F}_2 + (\nabla\phi)_2 \tag{5-4}$$

$$\frac{\partial v_1}{\partial t} = -(2\bar{\Omega}\times\bar{v})_\perp + \bar{F}_1 - [(\bar{v}\cdot\nabla)\bar{v}]_\perp \tag{5-5}$$

$$\frac{\partial \psi}{\partial t} = -\bar{v}\cdot\nabla\psi + \frac{1}{1-\kappa}(\dot{Q} - \nabla\cdot\bar{v}) \tag{5-6}$$

$$\frac{\partial T}{\partial t} = -\bar{v}\cdot\nabla T + \frac{T}{1-\kappa}(\mathring{Q} - \kappa\nabla\cdot\bar{v}) \tag{5-7}$$

$$\bar{v} = v_1\hat{e}_r + \bar{v}_2 \tag{5-8}$$

That is, the hydrostatic balance condition is used to simplify the equation of motion. The vertical part is integrated explicitly, and the vertical and horizontal parts added together to form a total $\bar{v}$. This approach has apparently not met with success, as it is nowhere reported in the literature thus far examined. It appears, however, that this approach is not equivalent to the approach taken in the Mintz-Arakawa and similar models, where the Hydrostatic equation is retained as a basic equation, instead of an auxiliary equation used to simplify the vertical motion equation.

Therefore, the approach that must be taken is to retain equation (5-4) but to compute a total $\bar{V}$ in such a way that the hydrostatic equation (5-3) is satisfied. In equation (5-6) and (5-7), the term $\nabla \cdot \bar{V}$ involves both V_1 and $\bar{V}_2$, as it is total $\bar{V}$ that must enter into the equations. Now, (5-3) relates the independent variables ψ, T in (5-6) and (5-7) Since $\bar{V}_2$ is given by (5-4), we then determine $\bar{V}_1$ to satisfy (5-3). Schematically

$$\bar{V}_2 = \bar{V}_2(\psi, T, \bar{V}_2, V_1) \qquad \text{(Equation of motion)}$$

$$\psi = \psi(\psi, V_1, \bar{V}_2) \qquad \psi\text{- equation (5-6)}$$

$$T = T(T, V_1, \bar{V}_2) \qquad T\text{- equation (5-7)}$$

$$g(\psi, T) = 0 \qquad \text{Hydrostatic equation}$$

What we must then do is to determine V_1 so that the last equation above is satisfied. We will perform the calculations in spherical coordinates. This will introduce some differences with the appearance of the corresponding formulas in cartesian coordinates. We will first require the expression for the ∇ operator in spherical coordinates.

$$\nabla = \hat{e}_r \frac{\partial}{\partial r} + \frac{\hat{e}_\theta}{r} \frac{\partial}{\partial \theta} + \frac{e_\varphi}{r \sin\theta} \frac{\partial}{\partial \varphi} \tag{5-9}$$

whence we define

$$\nabla_1 = \hat{e}_r \frac{\partial}{\partial r} \tag{5-10}$$

and

$$\nabla_2 = \frac{1}{r} \hat{e}_\theta \frac{\partial}{\partial \theta} + \frac{1}{r \sin\theta} \hat{e}_\varphi \frac{\partial}{\partial \varphi} \tag{5-11}$$

It is also convenient to regard this operator as the result of a gradient operation along a surface of constant r . In this case ∇_2 will also be written as ∇_r . Also, we have

$$\nabla \cdot \bar{V} = \left(\frac{\partial}{\partial r} + \frac{2}{r} \right) V_r + \nabla_2 \cdot \bar{V}_2 \tag{5-12}$$

$$= D_2 \bar{V}_r + \nabla_2 \cdot \bar{V}_2$$

Now, using (5-9) and (5-12) in equation (5-6) we have

$$\frac{\partial \psi}{\partial t} = -(V_1 \hat{e}_r + \bar{V}_2) \cdot \left(\frac{\partial \psi}{\partial r} \hat{e}_r + \nabla_2 \psi \right) + \frac{1}{1-K} \left(\dot{Q} - D_2 V_1 - \nabla_2 \cdot \bar{V}_2 \right)$$

$$= \frac{g}{RT} V_1 - V_2 \cdot \nabla_2 \psi + \frac{1}{1-K} \left(\dot{Q} - D_2 V_1 - \nabla_2 \cdot \bar{V}_2 \right) \tag{5-13}$$

We must now differentiate the entire equation above with respect to r in order to apply (5-3). This gives:

$$\frac{\partial}{\partial r} \left(\frac{\partial \psi}{\partial t} \right) = \frac{\partial}{\partial t} \left(\frac{\partial \psi}{\partial r} \right) = \frac{\partial}{\partial t} \left(\frac{-g}{RT} \right) = \frac{g}{RT^2} \frac{\partial T}{\partial t}$$

$$= \frac{g}{RT} \frac{\partial V_1}{\partial r} - \frac{g}{RT^2} \frac{\partial T}{\partial r} V_1 - \frac{1}{1-K} \left[\frac{\partial^2 V_1}{\partial r^2} - \frac{2}{r^2} V_1 + \frac{2}{r} \frac{\partial V_1}{\partial r} \right] \tag{5-14}$$

$$+ \frac{1}{1-K} \frac{\partial}{\partial r} \left[-(1-K) \bar{V}_2 \cdot \nabla_2 \psi - \nabla \cdot \bar{V}_2 + \dot{Q} \right]$$

Applying (5-3) allows us to equate the right side of (5-14) with $g/_{RT^2}$ times the right side of (5-7). We can also apply (5-9) and (5-12) to equation (5-7). We obtain:

$$-\frac{1}{1-\kappa}\frac{\partial^2 v_1}{\partial r^2} + \left[\frac{g(1-\kappa)}{RT} - \frac{2}{r}\right]\frac{1}{1-\kappa}\frac{\partial v_1}{\partial r} + \left[-\frac{g}{RT^2}\frac{\partial T}{\partial r}(1-\kappa) + \frac{2}{r^2}\right]v_1 + \frac{1}{1-\kappa}\frac{\partial}{\partial r}\left[(\kappa-1)\bar{v}_2\cdot\nabla_2\psi - \nabla_2\cdot\bar{v}_2 + \dot{Q}\right]$$

$$=$$

$$\frac{g}{RT^2}\left[-v_1\frac{\partial T}{\partial r} - \frac{\kappa T}{(1-\kappa)}\left(\frac{\partial v_1}{\partial r} + \frac{2v_1}{r}\right) + \frac{1}{1-\kappa}\left[(\kappa-1)\bar{v}_2\cdot\nabla_2 T - \nabla_2\cdot\bar{v}_2 + \dot{Q}\right]\right] \tag{5-15}$$

We may now collect the terms in V_1 to obtain an equation for V_1 which will cause the hydrostatic equation to be satisfied; namely:

$$\frac{\partial^2 V_1}{\partial t^2} + \left(\frac{2}{r} - \frac{g}{RT}\right)\frac{\partial v_1}{\partial r} + \left(-\frac{2}{r^2} + \frac{\kappa g}{RT}\right)V_1$$

$$=$$

$$-\frac{\partial}{\partial r}\left[(\kappa-1)\bar{v}_2\cdot\nabla_2\psi - \nabla_2\cdot\bar{v}_2 + \dot{Q}\right] + \left[(\kappa-1)\bar{v}_2\cdot\nabla_2 T - \nabla_2\cdot\bar{v}_2 + \dot{Q}\right] \tag{5-16}$$

This is a very complicated second-order differential equation for V_1. Although linear in V_1, it involves very cumbersome computations in order to find either the coefficients or the driving function (right side of (5-16)). The set of equations to be solved is now (5-4), (5-6), (5-7), (5-8) and (5-16). Because of the complications of (5-16) which would have to be solved at each grid point, we will abandon this set of equations in favor of a change to pressure coordinates and σ coordinates. The appearance of equation (5-16) is the deciding factor in the decision to change coordinates at this stage of program development. It should be noted, however, for future reference that if T is given, the homogeneous

portion of (5-16) might be solved analytically, which in some future system might be used to advantage. The computations for the forcing function would still be with us; and the computations would still have to be performed at each grid point and then transformed back to coefficients.

III.6 Change of Coordinates

This subsection concerns the form of the equations when the vertical coordinate r is replaced, first by the pressure and then by the dimensionless variable σ. In this section, the meaning of the variables Π and ψ differs within subsections. However, the meaning is clearly indicated within each subsection. Section III.6.1 recapitulates the equations derived in section III.5. Section III.6.2 deals with the implications of changing radial variable in spherical coordinates, which introduces some differences from the usual procedure on cartesian coordinates. Section III.6.3 deals with the pressure coordinate transformation; the independent variables being p, θ, φ, t. Sections III.6.4 deals with Sigma coordinates where the independent variables are σ, θ, φ, and t. The final form of the model is given in Section III.6.5; and section III.6.6 concerns Model Implementation considerations.

III.6.1 The (r, θ, φ, t) System

These equations are essentially a summary of the development given in III.5. Some detail has been added.

Define:

$$\psi(r,\theta,\varphi,t) = \ln p(r,\theta,\varphi,t) \qquad \text{(log pressure)}$$

$$\pi(\theta,\varphi,t) = \psi(r_e,\theta,\varphi,t) \qquad \text{(log surface pressure)}$$

The Model is then:

$$\frac{\partial \overline{V}_2}{\partial t} = -V_1 \frac{\partial \overline{V}_2}{\partial r} - (\overline{V}_2 \cdot \nabla_r)\overline{V}_2 - RT\nabla_r\psi - 2\Omega \cos(\hat{e}_r \times \overline{V}_2)$$
$$-2\Omega V_1 \sin\theta \, \hat{e}_\phi + \overline{F}_2 \tag{6-1}$$

$$\frac{\partial \pi}{\partial t} = \left[-\overline{V}_2 \cdot \nabla_r \pi + \frac{1}{1-k}\left(\dot{Q} - \nabla_r \cdot \overline{V}_2 - \frac{\partial V_1}{\partial r}\right)\right]_{r=r_e} \tag{6-2}$$

$$\frac{\partial T}{\partial t} = -V_1 \frac{\partial T}{\partial r} - \overline{V}_2 \cdot \nabla_r T \tag{6-3}$$
$$+ \frac{T}{1-k}\left[\dot{Q} - k\nabla_r \cdot \overline{V}_2 - k\frac{\partial V_1}{\partial r} - 2kV_1\right]$$

$$\frac{\partial \psi}{\partial r} = -\frac{g}{RT} \tag{6-4}$$

$$\frac{\partial^2 V_1}{\partial r^2} + \left(\frac{2}{r} - \frac{g}{RT}\right)\frac{\partial V_1}{\partial r} - \frac{2}{r}\left(\frac{1}{r} + \frac{kg}{RT}\right)V_1$$

$$= \frac{\partial \dot{Q}}{\partial r} - \frac{g}{RT}\dot{Q} - (1-k)\left(\frac{\partial \bar{V}_2}{\partial r} - \frac{\bar{V}_2}{r}\right)\cdot \nabla_r \gamma \tag{6-5}$$

$$-\nabla_r \cdot \frac{\partial \bar{V}_2}{\partial r} + \left(\frac{1}{r} + \frac{kg}{RT}\right)\nabla_r \cdot \bar{V}_2$$

with the supplemental Equations

$$p = e^{\gamma} \tag{6-6}$$

$$\rho = \frac{p}{RT} \tag{6-7}$$

the Initial Conditions required are

$$\bar{V}_2(r, \theta, \varphi, t_0)$$

$$\pi(\theta, \varphi, t_0)$$

$$T(r, \theta, \varphi, t_0)$$

and the Boundary Conditions are

$$T(r_e, \theta, \varphi, t) = T_s(\theta, \varphi) = T(r_e, \theta, \varphi, t_0)$$

$$\gamma(r_e, \theta, \varphi, t) = \pi(\theta, \varphi, t)$$

$$V_1(r_e, \theta, \varphi, t) = V_1(\infty, \theta, \varphi, t) = 0$$

III.6.2 Implications of Changing Radial Variable

The "horizontal del operator" (∇_r) is defined as

$$\nabla_r \equiv \frac{\hat{e}_\theta}{r}\frac{\partial}{\partial\theta} + \frac{\hat{e}_\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi} \tag{6-8}$$

for gradient and divergence. The notation $\nabla_r \times \overline{V}$ represents "vertical curl" of $\overline{V}$.

That is:

$$\nabla_r \times \overline{V} \equiv \left[\hat{e}_r \cdot \left\{(\nabla_r)\times\overline{V}\right\}\right]\hat{e}_r \neq (\nabla_r)\times\overline{V}$$

Using this notation the total derivative operators on a scalar field ϵ and a tangential velocity field $\overline{V}_2$ are

$$\frac{d\epsilon}{dt} = \frac{\partial\epsilon}{\partial t} + V_1\frac{\partial\epsilon}{\partial r} + \overline{V}_2\cdot(\nabla_r\epsilon) \tag{6-9}$$

$$\frac{d\overline{V}_2}{dt} = \frac{\partial\overline{V}_2}{\partial t} + V_1\frac{\partial\overline{V}_2}{\partial r} + (\overline{V}_2\cdot\nabla_r)\overline{V}_2 \tag{6-10}$$

where

$$V_1 = \frac{dr}{dt} \qquad \text{and}$$

$$(\bar{V}_2 \cdot \nabla_r)\bar{V}_2 = \frac{1}{2}\nabla_r(\bar{V}_2 \cdot \bar{V}_2) - \bar{V}_2 \times (\nabla_r \times \bar{V}_2)$$

To change the radial variable from r to say σ , then the "horizontal del operator", ∇_σ [understood to be applied on surfaces of constant σ, where $\sigma = \sigma(r, \theta, \varphi, t)$] is defined as

$$\nabla_\sigma \equiv \frac{\hat{e}_\theta}{\sigma}\frac{\partial}{\partial\theta} + \frac{\hat{e}_\varphi}{\sigma\sin\theta}\frac{\partial}{\partial\varphi} \tag{6-11}$$

for gradient and divergence and

$$\nabla_\sigma \times \bar{V}_2 \equiv \left[\hat{e}_r \cdot \{(\nabla_\sigma) \times \bar{V}\}\right]\hat{e}_r$$

The equations relating ∇_r to ∇_σ are;

$$\nabla_\sigma \epsilon = \frac{r}{\sigma}\nabla_r \epsilon + (\nabla_\sigma r)\frac{\partial\epsilon}{\partial r} \tag{6-12}$$

$$\nabla_\sigma \cdot \overline{V}_2 = \frac{r}{\sigma} \nabla_r \cdot \overline{V}_2 + (\nabla_\sigma r) \cdot \frac{\partial \overline{V}_2}{\partial r} \tag{6-13}$$

$$\nabla_\sigma \times \overline{V}_2 = \frac{r}{\sigma} \nabla_r \times \overline{V}_2 + (\nabla_\sigma r) \times \frac{\partial \overline{V}_2}{\partial r} \tag{6-14}$$

where ϵ is a scalar field that can be functionally represented as $\epsilon'(r,\theta,\varphi,t)$ or $\epsilon''(\sigma,\theta,\varphi,t)$. Similarly $\overline{V}_2$ is a tangential vector field.

The total derivative operator becomes

$$\frac{d\epsilon}{dt} = \frac{\partial \epsilon}{\partial t} + \frac{\partial \epsilon}{\partial \sigma}\frac{d\sigma}{dt} + \frac{\sigma}{r}\overline{V}_2 \cdot \nabla_\sigma \epsilon \tag{6-15}$$

$$\frac{d\overline{V}_2}{dt} = \frac{\partial \overline{V}_2}{\partial t} + \frac{\partial \overline{V}_2}{\partial \sigma}\frac{d\sigma}{dt} + \frac{\sigma}{r}(\overline{V}_2 \cdot \nabla_\sigma)\overline{V}_2 \tag{6-16}$$

where

$$(\overline{V}_2 \cdot \nabla_\sigma)\overline{V}_2 = \frac{1}{2}\nabla_\sigma(\overline{V}_2 \cdot \overline{V}_2) - \overline{V}_2 \times (\nabla_\sigma \times \overline{V}_2)$$

III.6.3 The (p, θ, ∅, t) System

The equations listed in III.6.1 were converted to pressure coordinates as a step in obtaining the model in σ - coordinates. The resulting equations do not seem to be computationally useful since there is difficu[illegible] in specifying boundary conditions.

In the conversion the more general equation

$$\frac{\partial \psi}{\partial t} = -V_1 \frac{\partial \psi}{\partial r} - \bar{V}_2 \cdot \nabla_r \psi + \frac{1}{1-k}\left(\dot{Q} - \nabla_r \cdot \bar{V}_2 - \frac{\partial V_1}{\partial r} - \frac{2}{r} V_1\right) \qquad (6\text{-}17)$$

is used in place of eqn (6-2).

Define

$$\omega \equiv \frac{dp}{dt}$$

The model is then

$$\frac{\partial \bar{V}_2}{\partial t} = -\omega \frac{\partial \bar{V}_2}{\partial p} - \frac{p}{r}(\bar{V}_2 \cdot \nabla_p)\bar{V}_2 - \frac{p}{r} g \nabla_p r$$
$$-2\Omega\cos\theta(\hat{e}_r \times \bar{V}_2) - 2\Omega\sin\theta\, V_1 \hat{e}_\phi + \bar{F}_2 \qquad (6\text{-}18)$$

$$\frac{\partial T}{\partial t} = -\omega \frac{\partial T}{\partial p} - \frac{p}{r} \bar{V}_2 \cdot \nabla_p T + T\left(\dot{Q} + \frac{k\omega}{p}\right) \tag{6-19}$$

$$\frac{\partial r}{\partial p} = -\frac{RT}{gp} \tag{6-20}$$

$$\frac{\partial \omega}{\partial p} = \frac{p}{r^2} \bar{V}_2 \cdot \nabla_p r - \frac{p}{r} \nabla_p \cdot \bar{V}_2 - \frac{2}{r} V_1 \tag{6-21}$$

$$\frac{\partial V_1}{\partial p} - \frac{2RT}{gpr} V_1 = \frac{(1-k)RT\omega}{gp^2} + \frac{RT}{gr} \nabla_p \cdot \bar{V}_2 + \frac{p}{r} \frac{\partial \bar{V}_2}{\partial p} \cdot \nabla_p r - \frac{RT}{gp} \dot{Q} \tag{6-22}$$

$$\rho = \frac{p}{RT} \tag{6-23}$$

III.6.4 The ($\sigma, \theta, \varphi, t$) System

The pressure coordinate equations of III.6.3 were converted to σ coordinate form. These equations are listed below.

Defining:

$$\pi \equiv p_s - p_T \tag{6-24}$$

where p_s is the surface pressure

and p_T is some constant reference pressure

$$\sigma \equiv \frac{p - p_T}{p_s - p_T} = \frac{p - p_T}{\pi} \tag{6-25}$$

$$\dot{\sigma} \equiv \frac{d\sigma}{dt} \tag{6-26}$$

The model becomes:

$$\begin{aligned}\frac{\partial \overline{V}_2}{\partial t} = &- \dot{\sigma} \frac{\partial \overline{V}_2}{\partial \sigma} - \frac{\sigma}{r}\left(\overline{V}_2 \cdot \nabla_\sigma\right)\overline{V}_2 - \frac{R\sigma^2 T}{r p} \nabla_\sigma \pi \\ &- \frac{\sigma g}{r} \nabla_\sigma r - 2\Omega \cos\theta \left(\hat{e}_r \times \overline{V}_2\right) - 2\Omega \sin\theta\, V_1\, \hat{e}_\phi + \overline{F}_2\end{aligned} \tag{6-27}$$

$$\frac{\partial \pi}{\partial t} = - \frac{\sigma}{r} \overline{V}_2 \cdot \nabla_\sigma \pi + \frac{\omega}{\sigma} - \frac{\pi \dot{\sigma}}{\sigma} \tag{6-28}$$

$$\frac{\partial T}{\partial t} = - \dot{\sigma} \frac{\partial T}{\partial \sigma} - \frac{\sigma}{r} \overline{V}_2 \cdot \nabla_\sigma T + T\left(\dot{Q} + \frac{\kappa \omega}{p}\right) \tag{6-29}$$

$$\frac{\partial r}{\partial \sigma} = - \frac{\pi R T}{g p} \tag{6-30}$$

$$\frac{\partial \omega}{\partial \sigma} = \frac{\sigma \pi}{r^2} \bar{V}_2 \cdot \nabla_\sigma r + \frac{\sigma^2 \pi}{r^2} \frac{RT}{gp} \bar{V}_2 \cdot \nabla_\sigma \pi - \frac{\pi \sigma}{r} \nabla_\sigma \cdot \bar{V}_2 + \frac{\sigma^2}{r} \frac{\partial \bar{V}_2}{\partial \sigma} \cdot \nabla_\sigma \pi - \frac{2}{r} V_1 \tag{6-31}$$

$$\frac{\partial \dot{\sigma}}{\partial \sigma} - \frac{\dot{\sigma}}{\sigma} = \frac{\sigma}{r^2} \bar{V}_2 \cdot \nabla_\sigma r - \frac{\sigma}{r} \nabla_\sigma \cdot \bar{V}_2 - \frac{2}{r} V_1 - \frac{\omega}{\pi \sigma} \tag{6-32}$$

$$\frac{\partial V_1}{\partial \sigma} - \frac{2 R \pi T}{g p r} V_1 = \frac{(1-k) R T \omega \pi}{g p^2} + \frac{R T \pi \sigma}{g p r} \nabla_\sigma \cdot \bar{V}_2 + \frac{\sigma}{r} \frac{\partial \bar{V}_2}{\partial \sigma} \cdot \nabla_\sigma r - \frac{R T \pi}{g p} \dot{Q} \tag{6-33}$$

With the supplemental equations

$$p = \pi \sigma + p_T \tag{6-34}$$

$$\rho = \frac{p}{RT} \tag{6-35}$$

The required initial conditions are;

$$\bar{V}_2 (\sigma, \theta, \varphi, t_o)$$

$$\pi (\theta, \varphi, t_o)$$

$$T (\sigma, \theta, \varphi, t_o)$$

and the boundary conditions are;

$$T(1, \theta, \varphi, t) = T_s(\theta, \varphi) = T(1, \theta, \varphi, t_o)$$

$r(1, \theta, \varphi, t) = r_e$ or in the more general case

$$= r_e(\theta, \varphi)$$

$$\omega(1, \theta, \varphi, t) = 0$$

$$\dot{\sigma}(1, \theta, \varphi, t) = \dot{\sigma}(0, \theta, \varphi, t) = 0$$

$$V_1(1, \theta, \varphi, t) = 0$$

The equations used by RAND, Ref (1), can be cast in our notation as;

$$\frac{\partial \bar{V}_2}{\partial t} = -\dot{\sigma}\frac{\partial \bar{V}_2}{\partial \sigma} - \frac{\sigma}{r_e}(\bar{V}_2 \cdot \nabla_\sigma)\bar{V}_2 - \frac{\sigma^2 RT}{r_e p}\nabla_\sigma \pi$$

$$-\frac{\sigma}{r_e} g \nabla_\sigma r - 2\Omega \cos\theta(\hat{e}_r \times \bar{V}_2) + \bar{F}_2 \qquad (6\text{-}36)$$

$$\frac{\partial \pi}{\partial t} = \frac{\omega}{\sigma} - \pi\frac{\dot{\sigma}}{\sigma} - \frac{\sigma}{r_e}\bar{V}_2 \cdot \nabla_\sigma \pi \qquad (6\text{-}37)$$

$$\frac{\partial T}{\partial t} = -\dot{\sigma}\frac{\partial T}{\partial \sigma} - \frac{\sigma}{r_e}\bar{V}_2 \cdot \nabla_\sigma T + \frac{\dot{H}}{c_p} + \frac{kT\omega}{p} \qquad (6\text{-}38)$$

where $\dot{Q} = \dfrac{\dot{H}}{c_p T}$

$$\frac{\partial \phi}{\partial \sigma} = -\frac{\pi RT}{p} \qquad (6\text{-}39)$$

where $\phi = g(r - r_e)$

$$\frac{\partial \omega}{\partial \sigma} = -\frac{\pi \sigma}{r_e} \nabla_\sigma \cdot \overline{V}_2 + \frac{\sigma^2}{r_e} \frac{\partial \overline{V}_2}{\partial \sigma} \cdot \nabla_\sigma \pi \tag{6-40}$$

$$\frac{\partial \dot{\sigma}}{\partial \sigma} - \frac{\dot{\sigma}}{\sigma} = -\frac{\sigma}{r_e} \nabla_\sigma \cdot \overline{V}_2 - \frac{\omega}{\pi \sigma} \tag{6-41}$$

The term $\frac{\sigma}{r_e}(\overline{V}_2 \cdot \nabla_\sigma)\overline{V}_2$ does not appear to be in the RAND equations, but is clearly included in the equations presented by Langlois & Kwok, Ref (3) . We are assuming that we have misinterpreted the RAND notation.

Equations (6-27) thru (6-33) have been developed entirely in spherical coordinates.

Note that the following approximations applied to these equations will generate the equations (6-36) thru (6-41).

$$\frac{1}{r} \rightarrow \frac{1}{r_e}$$

$$\frac{1}{r^2} \rightarrow 0$$

$$\frac{V_1}{r} \rightarrow 0$$ except in horizontal momentum

equation (6-27) where $V_1 \rightarrow 0$

Equation (6-33) is not needed since all occurrences of V_1 in the other equations have been eliminated.

III.6.5 Final Form of Model

To obtain the form of equations (6-34) to (6-41) most suitable for a spectral solution, use equation (6-37) to eliminate the variable ω from the problem. Then eqn (6-41) becomes

$$\frac{\partial \dot{\sigma}}{\partial \sigma} = -\frac{\sigma}{r_e} \nabla_\sigma \cdot \bar{V}_2 - \frac{\sigma}{\pi r_e} \bar{V}_2 \cdot \nabla_\sigma \pi - \frac{1}{\pi} \frac{\partial \pi}{\partial t} \qquad (6\text{-}42)$$

Integrate (6-42) over σ from 1 to 0 using the boundary condition $\dot{\sigma} = 0$ at $\sigma = 1$ & $\sigma = 0$.

$$\frac{1}{\pi} \frac{\partial \pi}{\partial t} = \int_1^0 \left(\frac{\sigma}{r_e} \nabla_\sigma \cdot \bar{V}_2 + \frac{\sigma}{\pi r_e} \bar{V}_2 \cdot \nabla_\sigma \pi \right) d\sigma \qquad (6\text{-}43)$$

Then define

$$\gamma = \ln \pi$$

$$\phi = g(r - r_e)$$

and let

$$p_T = 0$$

and let ∇_σ include $\frac{\sigma}{r_e}$ as a factor to obtain the model;

$$\frac{\partial \bar{V}_2}{\partial t} + \dot{\sigma}\frac{\partial \bar{V}_2}{\partial \sigma} + (\bar{V}_2 \cdot \nabla_\sigma)\bar{V}_2 + 2\Omega \cos\theta (\hat{e}_r \times \bar{V}_2) + \nabla_\sigma \phi + R\sigma\left(\frac{T}{\sigma}\right)\nabla_\sigma \gamma = \bar{F}_2 \tag{6-44}$$

$$\frac{\partial\left(\frac{T}{\sigma}\right)}{\partial \sigma} + \dot{\sigma}\frac{\partial\left(\frac{T}{\sigma}\right)}{\partial \sigma} + \bar{V}_2 \cdot \nabla_\sigma\left(\frac{T}{\sigma}\right) = \frac{\dot{H}}{c_p \sigma} - \frac{\dot{\sigma}}{\sigma}\left(\frac{T}{\sigma}\right) + k\left(\frac{T}{\sigma}\right)\left[\frac{\partial \gamma}{\partial t} + \frac{\dot{\sigma}}{\sigma} + \bar{V}_2 \cdot \nabla_\sigma \gamma\right] \tag{6-45}$$

$$\frac{\partial \phi}{\partial \sigma} = -R\left(\frac{T}{\sigma}\right) \tag{6-46}$$

$$\frac{\partial \psi}{\partial t} = \int_1^0 \left(\nabla_\sigma \cdot \bar{V}_2 + \bar{V}_2 \cdot \nabla_\sigma \psi \right) d\sigma \tag{6-47}$$

$$\frac{\partial \dot{\sigma}}{\partial \sigma} = - \nabla_\sigma \cdot \bar{V}_2 - \bar{V}_2 \cdot \nabla_\sigma \psi - \frac{\partial \psi}{\partial t} \tag{6-48}$$

The supplemental equations are:

$$\phi = g(r - r_e) \tag{6-49}$$

$$\psi = \ln \pi \tag{6-50}$$

$$p = \pi \sigma \tag{6-51}$$

$$\bar{F}_2 = \begin{cases} K_1 |\bar{V}_2| \bar{V}_2 + K_2 \dfrac{\partial^2 \bar{V}_2}{\partial r^2} & \text{in layer 1} \\ K_3 \dfrac{\partial^2 \bar{V}_2}{\partial r^2} & \text{in layer 2} \\ 0 & \text{in higher layers} \end{cases} \tag{6-52}$$

$$\begin{aligned} \dot{H} &= K_4 \sin^2(\theta - K_5) \operatorname{Max}\left[\sin(\Omega t - \varphi + K_6),\ 0\right] e^{K_7 e^{K_8 (r - r_e)}} \\ &\quad + K_9 T_s^4 e^{K_{10} e^{K_{11}(r - r_e)}} \end{aligned} \tag{6-53}$$

The required initial conditions are:

$$\bar{V}_2(\sigma, \theta, \varphi, t_0)$$

$$T(\sigma, \theta, \varphi, t_0)$$

$$\psi(\theta, \varphi, t_0)$$

The boundary conditions are:

$$\phi(1, \theta, \varphi, t) = 0$$

$$\dot{\sigma}(1, \theta, \varphi, t) = \dot{\sigma}(0, \theta, \varphi, t) = 0$$

$$T(1, \theta, \varphi, t) = T_s(\theta, \varphi)$$

III.6.6 Considerations for Implementing the Model

The radial expansion functions to be used in this implementation are Legendre polynomials. They are orthogonal over the interval (-1, 1). The variable S is a mapping of σ that is used as the argument of the Legendre polynomials in the implementation. Computations are performed at five values of σ (computational layers). Figure 6-1 illustrates the relationships between σ, S, ρ nominal, h nominal at the five computational layers.

Figure (6-1) Radial Structure

LAYER NUMBER	S	σ	p_{NOM}	h_{NOM}
N/A	$S=1$	$\sigma=0$	0	∞
5	$S=S_5 \cong .906$	$\sigma=\sigma_5 \cong .049$	49 mb	20.6 km
4	$S=S_4 \cong .538$	$\sigma=\sigma_4 \cong .243$	243 mb	10.2 km
3	$S=S_3=0$	$\sigma=\sigma_3 \cong .525$	525 mb	5.2 km
2	$S=S_2 \cong -.538$	$\sigma=\sigma_2 \cong .807$	807 mb	1.8 km
1	$S=S_1 \cong -.906$	$\sigma=\sigma_1=1$	1000 mb	0
N/A	$S=-1$	N/A	N/A	N/A

The values s_i are the roots of the equation

$$P_s(s) = 0$$

and were chosen because they are the sample points for Gaussian integration. The actual mapping relationship is given by

$$s = 1 + (s_1 - 1)\,\sigma \qquad \text{(6-54)}$$

For the implementation of the "horizontal" del operator, where ∇_σ includes the factor $\frac{\sigma}{r_e}$, we have

$$\nabla_\sigma = \frac{\hat{e}_\theta}{r_e}\frac{\partial}{\partial\theta} + \frac{\hat{e}_\varphi}{r_e \sin\theta}\frac{\partial}{\partial\varphi}$$

Then the "horizontal" gradient, divergence and the vertical curl of scalar and vector spherical harmonics are given by:

$$\nabla_\sigma\left[f(\sigma)\,Y_L^M\right] = \alpha_L \frac{L}{r_e} f(\sigma)\,T^M_{L\,L+1} + \beta_L \frac{L+1}{r_e} f(\sigma)\,T^M_{L\,L-1} \qquad \text{(6-55)}$$

$$\nabla_r \cdot \left[f(r)\, T^{M}_{L\,L-1} \right] = -\beta_L \frac{(L+1)}{r_e} f(r)\, Y^{M}_{L} \tag{6-56}$$

$$\nabla_r \cdot \left[f(r)\, T^{M}_{L\,L} \right] = 0 \tag{6-57}$$

$$\nabla_r \cdot \left[f(r)\, T^{M}_{L\,L+1} \right] = -\alpha_L \frac{L}{r_e} f(r)\, Y^{M}_{L} \tag{6-58}$$

$$\nabla_r \times \left[f(r)\, T^{M}_{L\,L-1} \right] = 0 \tag{6-59}$$

$$\nabla_r \times \left[f(r)\, T^{M}_{L\,L} \right] = i\left[\alpha_L \frac{L}{r_e} f(r)\, T^{M}_{L\,L-1} - \beta_L \frac{(L+1)}{r_e} f(r)\, T^{M}_{L\,L+1} \right] \tag{6-60}$$

$$\nabla_r \times \left[f(r)\, T^{M}_{L\,L+1} \right] = 0 \tag{6-61}$$

III.7 Implementation Functional Flow

The solution of equations (6-44) to (6-48) involves integrations over time and sigma. The equations are all formed in physical space and all integrations are performed in spectral space. The sequence of integrations needed to solve the equations is arranged as a "cascade" process. This is done to provide, wherever possible, the latest values for quantities appearing in an equation to be integrated.

This technique, when applied to a second order linear oscillator of the form

$$\frac{d^2 x}{dt^2} + c^2 x = 0$$

has the stability properties of a centered difference scheme. A rigorous stability analysis cannot be made of the technique applied to the highly nonlinear system given in equations (6-44) to (6-48). It is, however, hoped that the stability properties demonstrated on the linear system will be retained. Furthermore, unlike centered difference schemes, all possible up-to-date information is used in the solution of each equation. It is therefore hoped that the problem of diverging solutions generated by alternate time steps mentioned by Haltiner in his discussion of centered difference schemes (reference 4) can be avoided.

The variables are obtained in the following order: $\phi, \bar{\sigma}, \psi, \dot{\sigma}, \overline{T}$. Several factors influence the computational sequence.

Note that both $\partial\psi/\partial t$ and $\nabla_\sigma \psi$ are required in the equations. We will thus have to integrate over sigma to obtain $\partial\psi/\partial t$ from equation (6-47) and then integrate $\partial\psi/\partial t$ over time to obtain ψ for forming $\nabla_\sigma \psi$. Since ψ is independent of σ, we solve for $\frac{\partial\psi}{\partial t}$ and ψ only during the computations for the first σ layer.

Also, note that $\nabla_\sigma \psi$ and $\partial\psi/\partial t$ both appear in equation (6-48), the equation for $\dot{\sigma}$. Equation (6-47), the equation for $\partial\psi/\partial t$, also contains $\nabla_\sigma \psi$. In order to have consistent ψ-dependent quantities in equation (6-48), we iterate on equation (6-47). First equation (6-47) is formed and integrated for $\partial\psi/\partial t$, which is in turn integrated for ψ. $\nabla_\sigma \psi$ is computed and equation (6-47) is re-evaluated and integrated for $\partial\psi/\partial t$. The latest value of $\nabla_\sigma \psi$ and the value of $\partial\psi/\partial t$ are then used in equation (6-48) to compute $\partial\dot{\sigma}/\partial\sigma$. Thus the value of $\nabla_\sigma \psi$ used in the equation for $\dot{\sigma}$ is the one used to compute the value of $\partial\psi/\partial t$ which also appears in equation (6-47).

In equation (6-45), the equation for T/σ, there is a triple product, namely $(\frac{T}{\sigma})\, \bar{v}_2 \cdot \nabla_\sigma \psi$. This is the only triple product appearing in the equations. If aliasing errors are to be avoided for this triple product, a much finer physical grid spacing is necessary than is required for any of the other terms appearing in the equations. Rather than accept the computational penalties imposed by the finer grid, the following computational scheme is used. The product $\bar{v}_2 \cdot \nabla_\sigma \psi$ is formed in physical space, transformed to spectral space and back to physical space before being incorporated in the equation for T/σ. This assures, given proper grid spacing, that truncation of the spectral expansion is done to avoid aliasing.

The sequence of steps for one complete integration of equations (6-44) to (6-48) can be outlined as follows:

- o Form $\partial\phi/\partial\sigma$ and integrate for ϕ to form $\nabla_\sigma \phi$
- o Form $\partial\bar{v}/\partial t$ and integrate for $\bar{v}$. Form $\nabla_\sigma \cdot \bar{v}$
- o Form the integral equation for $\partial\psi/\partial t$ using the $\bar{v}$ and $\nabla_\sigma \cdot \bar{v}$ just obtained and integrate for $\partial\psi/\partial t$
- o Integrate $\partial\psi/\partial t$ for ψ and form $\nabla_\sigma \psi$
- o Form the integral equation for $\partial\psi/\partial t$ using the new $\nabla_\sigma \psi$ and integrate for $\partial\psi/\partial t$
- o Form the equation for $\partial\dot{\sigma}/\partial\sigma$ and integrate for $\dot{\sigma}$
- o Form the product $\bar{v} \cdot \nabla_\sigma \psi$ and perform the transforms from physical space to spectral space to physical space
- o Form $\partial(T/\sigma)/\partial t$ and integrate for T/σ

References for Section III.

(1) Gates, W. L. et.al., A Documentation of the Mintz-Arakawa Two-Level Atmoshperic General Circulation Model. Report R-877-ARPA, The RAND Corp; Santa Monica, CA. (Dec. 1971)

(2) Bjorklund, S. Global Weather Modeling with Vector Spherical Harmonics, Report No. 1. IBM Corporation, Federal Systems Division, Gaithersburg, MD. (Nov. 1972)

(3) Langlois, W.E. and Kwok, H.C.W., Description of the Mintz-Arakawa Numerical General Circulation Model. Technical Report No. 3., Numerical Simulation of Weather and Climate. Department of Meteorology, University of California, Los Angeles, CA. (Feb. 1969)

(4) Haltiner, G. J, Numerical Weather Prediction, Navy Weather Research Facility, Norfolk, VA. (1968).

IV Computational Considerations of the Transform Method

This section deals with the computational requirements imposed by the Transform Method. Most of these requirements are imposed to avoid "aliasing" in transforming from physical grids to spectral coefficients. In subsection IV.1 aliasing is defined. General solutions are outlined in IV.2, and particular results for VSH modeling are presented in subsection IV.3.

IV.1 Aliasing

Aliasing arises when dealing with functions that are expanded in series of orthogonal functions such as Fourier Series, orthogonal polynomials, scalar or vector spherical harmonics, etc. We will call such functions "Harmonics" in a general sense. Suppose a function F(x) of a variable x is expanded in series of harmonics, which we will denote . We assume the are orthogonal over some interval (O,L) in x; then:

$$f(x) = \sum_{n=0}^{\infty} c_n \varphi_n(x) \tag{1-1}$$

$$c_n = \int_0^L f(x)\, \varphi_n(x)\, dx \tag{1-2}$$

where

$$\int_0^L \varphi_n(x)\, \varphi_m(x) = \delta_{nm} \tag{1-3}$$

In order to obtain the values of f(x), we can evaluate equation (1-1). Conversely, given all the values of f(x), we can obtain the values of all the coefficients c_n.

A case of practical interest arises when f(x) can be represented in a finite number of expansion coefficients, i.e., when

$$f(x) = \sum_{n=0}^{N-1} c_n \varphi_n(x) \tag{1-4}$$

In this case, we will say that f(x) is band-limited. It is well known that in this case, to evaluate the N coefficients C_n we do not in general require the values of F(x) at all x. Instead, given a certain number of samples of f(x) (not necessarily equidistant) we can obtain the coefficients c_n. These coefficients C_n will reconstruct the function f(x) exactly (by equation 1-1). The number of samples required depends upon the form of the f(x).

On the other hand, if we are given less than the above critical number of samples, the C_n that are obtained from eg. (1-2) will not be correct. The "lower frequency" coefficients (n= Ø, 1,2...) will be in error. This condition is known as aliasing. It applies to all spectral modeling activities since they all work with truncated series, i.e., functions that are forced to be band-limited.

The value of N needed to represent f(x) in 1-4 will be called the Harmonic Content of f(x).

IV.2 Avoiding Aliasing Effects

In order to avoid aliasing completely, two general approaches can be followed:

(1) Ensure that enough data samples are in hand to compute coefficients accurately,

(2) Guarantee that all functions arising in modeling activities are properly band-limited.

Both approaches must be followed in practical modeling. In general, most problems arise in nonlinear operations. For example, if two sinusoidal terms of a given frequency are multiplied together, the result is a sinusoidal term of double the frequency. In general, nonlinear operations raise the harmonic content, in the sense that more terms in the series of

type (1-4) are necessary to express the result exactly, even if the original terms were band-limited. For example, a division introduces an infinite harmonic content.

Since the transform method consists of transforming variables to a physical grid, and performing the nonlinear operation on this grid, the general outline of the procedure that must be followed in spectral models with the transform method is:

(1) Tailor the computational procedure such that nonlinearities which yield a harmonic content higher than the model's truncation limits are minimized. For example, avoid division, reduce number of products, etc.

(2) Once the above is done, evaluate each non-linear operation to determine the maximum harmonic content of all the nonlinear operations.

(3) Determine the size and spacing of the physical grid such that the maximum harmonic content introduces no aliasing.

When the procedure is performed, we must know the original harmonic content, the types of nonlinearity, and the effect of each nonlinearity on the harmonic content. For example, the nonlinearities introduced by the advection term $(\bar{V}.\nabla)\ \bar{v}$ are known from its analytic formula, Section I.

IV.3 Avoiding Aliasing in VSH Modeling

Two general types of functions appear in VSH modeling: polynomials and circular functions. In addition, the radial functions chosen may be of another type. But for the modeling discussed in Section III, we have specifically:

1) Circular functions $e^{im\varphi}$ in the vector and scalar spherical harmonics.

2) Associated Legendre polynomials P_ℓ^m in vector and scalar sperical harmonics.

3) Legendre polynomials $P_\ell^0 = P_\ell$ in radial functions.

All of these functions are orthogonal, and we can find the coefficients by applying (1-2). It is well known that:

1) For polynomial type functions, the best method of reconstruction is Gaussion integration. With this method, a polynomial of degree $K \leq 2N-1$ can be integrated exactly over an interval (-1,1) with N sample points located at the zeros of the Legendre polynomial of degree N.

2) Circular functions do not use Gaussion integration. The K^{th} coefficient, $K \leq (N-1)/2$ can be recovered with N equally spaced points over the interval equal to the period of the fundamental, $(0, 2\pi)$.

using these basic relationships, we can derive the rules which are used to compute grid sizes for the transformation. The rules are given for each specific function type below. These are based on a maximum 'harmonic generating' mechanism equivalent to the product of two variables.

a. Radial functions $P_n(s)$, $N = 0,1,2, \ldots N-1$

$$-1 \leq s \leq 1$$

The number of layers N_r required is given by

$$N_r \geq \frac{3N-2}{2} \qquad (3-1)$$

For example if $N = 4$ (i.e. we use $P_0(s), \ldots, P_3(s)$)
then $N_r \geq (3.4-2)/2 = 5$

Points have a Gaussian spacing.

b. Colatitude functions $P_\ell^m(\cos\theta)$

$$\ell = 0,1,2,\ldots L-1$$
$$\theta \leq o \leq \pi;\ -1 \leq \cos\theta \leq 1$$

The number of θ-points is given by

$$N_\theta > \frac{3L-1}{2} \qquad (3\text{-}2)$$

For example if $L=10$, $N_\theta \geq 15$

(The model uses $N_\theta = 16$ for symmetry reasons)

Points have a Gaussian spacing.

c. Longitude functions $e^{iM\varphi}$

$$|M| \leq 0,1,2,\ldots M-1$$

The number of φ-points is given by

$$N\varphi \geq 4M-L$$

For example, if $M=10$, $N_\varphi \geq$ 39

The model uses 40 for symmetry reasons.)
The points must be equally spaced on $(0, 2\pi)$.
If a Fast Fourier transform were used, the above would need some modification.

A few additional comments are in order. In the model equations in Section III, there are two divisions. One is given by $\dot{\sigma}/\sigma$ terms. This case is acceptable, because when $\dot{\sigma}$ is developed in Legendre polynomials, then σ is a factor of $\dot{\sigma}$ and therefore the division is exact. The other division, in the forcing fields, will in fact introduce some aliasing. In general any forcing field or boundary condition causes aliasing. If this aliasing proves empirically troublesome, then the grid size will have to be expanded, which means that additional coefficients would be used.

One triple product is present in the model in Section III. Aliasing in this product is avoided by using the product computations twice and truncating the intermediate result.

IV.4 Verification of Transform Method

A set of utility programs for manipulating scalar and vector spherical harmonics have been programmed in the APL language. These include:

- Clebsh - Gordan coefficient computation
- Racah, 3-J and 6-J symbol computation
- Product of Scalar harmonics $(Y_L^M \; Y_{L'}^{M'})$
- Scalar times vector harmonics $(Y_{L'}^{M'},\; T_{JL}^{M})$
- Vector harmonic dot product $(T_{JL}^{M} \cdot T_{J'L'}^{M'})$
- Vector harmonic cross product $(T_{JL}^{M} \times T_{J'L'}^{M'})$
- Gradient & Laplacian of scalar harmonics
- Divergence, Curl, Laplacian of vector harmonics
- Advection $(T_{JL}^{M} \cdot \nabla)\; T_{J'L'}^{M'}$

These routines are coded to operate entirely on harmonic expansion coefficients.

The programming system computes the expressions for the derivatives of the forecast variables in physical space. The derivatives (physical) are then expanded in spectral coefficients. Each term in the expansion of the physical derivative is the derivative of the corresponding term in the expansion of the physical variable.

To test the code that does the spectral to physical transformations and to verify the validity of the transform approach, a series of comparison runs were made.

In these runs the scalar and vector fields in the model were initialized to several different sets of coefficient values. The program system was then run through one cycle that included the following steps:

a. Transform variable fields to physical space
b. Compute spatial derivatives in coefficient space
c. Transform results of (b) to physical space
d. Form a comprehensive set of non-linear terms in physical space
e. Transform results of (d) to spectral space.

The output of (e) was compared to the results of APL runs where the various non-linearities were formed using the fundamental relationships of vector and scalar spherical harmonics on a term by term basis on the initial field coefficient values. Both the program system and APL operate in double precision (approx. 16 decimal digits resolution). In all cases there was agreement to the thirteenth digit.

V. Radial Functions Analysis

V.I Introduction

Expansion of vector or scalar variables in vector or scalar spherical harmonics allows us to separate the variables of a partial differential equation in the sense that the angular part can be integrated out, leaving a system of partial differential equations in the harmonic coefficients. If the problem is not time-dependent, the equations become ordinary differential equations in r but in the case of a time-dependent problem, the spectral coefficients are functions of r and t. By expanding the coefficients again, in some set of suitable functions of r, the problem can be reduced to a set of ordinary differential equations in time. The approach could be further extended to expand the remaining coefficients in functions of time, leaving us with a purely algebraic problem. As an example, consider the partial differential equation

$$\frac{\partial \bar{v}}{\partial t} = \nabla \times \bar{v} \; ; \qquad \bar{v} = \bar{v}(r, \vartheta, \varphi, t) \tag{1-1}$$

in order to solve this equation, we begin by expanding the vector field $\bar{v}$ in vector spherical harmonics:

$$\bar{v} = \sum_{M,L} a_L^M(r,t)\, A_L^M(\vartheta,\varphi) + b_L^M(r,t)\, B_L^M(\vartheta,\varphi) + c_L^M(r,t)\, C_L^M(\vartheta,\varphi)$$

using the relations for curl gives (1-2)

$$\sum_{M,L} \frac{\partial a_L^M}{\partial r} A_L^M + \frac{\partial b_L^M}{\partial t} B_L^M + \frac{\partial c_L^M}{\partial t} C_L^M = \sum_{M,L} \frac{i\gamma_L}{r} c_L^M A_L^M + i D_1 c_L^M B_L^M + \left(-\frac{i\gamma_L}{r} a_L^M + i D_1 b_L^M\right) C_L^M$$

(1-3)

and equating the coefficients gives the set of partial differential equations

$$\frac{\partial a_L^M}{\partial t} = \frac{i\,\gamma_L}{r}\, c_L^M \tag{1-4}$$

$$\frac{\partial b_L^M}{\partial t} = i D_1\, c_L^M \tag{1-5}$$

$$\frac{\partial c_L^M}{\partial t} = -i\frac{\gamma_L}{r}\, a_L^M + i D_1 b_L^M \tag{1-6}$$

Now, the a_L^M , b_L^M , and c_L^M are functions of r and t. In this particular case, the exact functional form can be determined by formal separation of variables in equations (1-4) through (1-6). But suppose, as is the case with more complicated problems, that we are unaware of the exact functional form. Then we can illustrate the expansion process by choosing a set of radial functions and expanding the a_L^M , b_L^M , and c_L^M coefficients. Suppose we choose Fourier Series as basis functions (because of their simplicity). Let us therefore, expand in a truncated series of 2N Fourier terms

$$a_L^M(r,t) = \sum_{K=-N}^{N} a_{LK}^M(t)\, e^{ikr} \tag{1-7}$$

$$b_L^M(r,t) = \sum_K b_{LK}^M(t)\, e^{ikr} \tag{1-8}$$

$$c_L^M(r,t) = \sum_K c_{LK}^M(t)\, e^{ikr} \tag{1-9}$$

That is, the new unknown coefficients are functions of time alone. The coefficient's radial dependence is now explicit, so

$$\frac{\partial a_L^M(r,t)}{\partial t} = \sum_K \frac{da_{LK}^M(t)}{dt}\, e^{ikr} \tag{1-10}$$

with similar relations for b_L^M and c_L^M . Then using (1-7) - (1-10) in (1-4) - (1-6) we find

$$\sum_K \frac{da^M_{LK}}{dt} e^{ikr} = \sum_K \frac{i\gamma_L}{r} c^M_{LK}(t) e^{ikr} \tag{1-11}$$

$$\sum \frac{db^M_{LK}}{dt} e^{ikr} = \sum i\left(ik - \frac{1}{r}\right) c^M_{LK}(t) e^{ikr} \tag{1-12}$$

$$\sum \frac{dc^M_{LK}}{dt} e^{ikr} = \sum \frac{i\gamma_L}{r} a^M_{LK}(t) e^{ikr} + i\left(ik - \frac{1}{r}\right) b^M_{LK}(t) e^{ikr} \tag{1-13}$$

Or, as the e^{ikr} functions are orthogonal,

$$\frac{da^M_{LK}}{dt} = \frac{i\gamma_L}{r} c^M_{LK} \tag{1-14}$$

$$\frac{db^M_{LK}}{dt} = i\left(ik - \frac{1}{r}\right) c^M_{LK} \tag{1-15}$$

$$\frac{dc^M_{LK}}{dt} = \frac{i\gamma_L}{r} a^M_{LK} + i\left(ik - \frac{1}{r}\right) b^M_{LK} \tag{1-16}$$

For any given r, equations (1-14) - 1-16) are now ordinary differential equations in time; each value of M and L yields N equations, each of the type (1-14) - (1-16). This set happens to be uncoupled because the example chosen is very simple.

A better choice of radial functions in this case is the set of functions e^{ikr}/r since they happen to be eigenfunctions of the radial operation D_1 , with eigenvalue (ik). In this case we would have obtained

$$\dot{a}^{M}_{LK} = \frac{L k_c}{r} c^{M}_{LK} \tag{1-17}$$

$$\dot{b}^{M}_{LK} = -K c^{M}_{LK} \tag{1-18}$$

$$\dot{c}^{M}_{LK} = \frac{L k_c}{r^2} a^{M}_{LK} - K b^{M}_{LK} \tag{1-19}$$

In fact, the appropriate radial functions in this case are spherical Bessel functions (1). This can be ascertained either by substitution or by formal separation of variables in (1-4) through (1-6). However, the intent here is to illustrate the procedure and not to particularize results.

Although the Fourier Series chosen do not lead to as simple a form as the functions e^{ikr}/r, they have the advantage of behaving well under product-type nonlinearities, since products can be replaced by sums.

The preceding discussion illustrates some of the desirable properties of radial function sets. These properties will be discussed in more detail in the following subsection. Subsection V.3 will review some candidates in more detail, in the light of the desirable properties discussed in V.2.

V.2 Desirable Properties of Radial Functions

V.2.1 Orthogonality

It is first of all desirable that the members of the set of radial functions chosen (henceforth "candidates") be mutually orthogonal. This is obviously advantageous from the analytic point of view. From the computational point of view, orthogonality is desirable because it permits the calculation of expansion coefficients without "aliasing" effects (see Section IV). Also, the computation of expansion coefficients is usually simplified by the orthogonality condition.

It is also desirable that the functions all be orthogonal over the same interval and, for meteorological purposes, that the interval map conveniently into the radial dimension of the atmosphere (as a counter-example, Bessel functions are orthogonal only over zeros of adjoining functions).

V.2.2 Derivative Properties

The radial differential operator which always occurs in VSH expansions is

$$D_L \equiv \left(\frac{\partial}{\partial r} + \frac{L}{r}\right) \quad ; \quad L = \ldots -2, -1, 0, 1, 2, \ldots \qquad (2\text{-}1)$$

Thus, it is desirable that the candidates have the closure property under this operator, i.e., that the operator applied to a member of the set yield a linear combination of members of the set. That is, if φi is a member of the candidate set,

$$D_L' \varphi_L(r) = \sum_{K(L,L)} C_K \varphi_K(r) \tag{2-2}$$

where the limits of the summation depend on the set of functions and L. The C_K are coefficient which also depend on the set of functions and L. This property permits us to analyze the "harmonic content" of the derivative expressions and is therefore useful computationally in estimating aliasing. As an example, if φk is a polynomial of degree K in r, its derivative is a polynomial of degree K-1. It csn obviously be expressed as sum of at most K-1 φn's, $n \leq K-1$.

If we only require one term in (2-2) then φi is an eigenfunction of D_L. Usually it is not possible to find simultaneous eigenfunctions of all the D_L's appearing in a given problem, so we must approximate by using a truncated expansion.

If the candidate is not closed under the D operator, then at least we should require closure under the d/dr operation to estimate aliasing effects.

V.2.3 Product Rule

In many applications we must deal with product terms, e.g., $(\bar{V}.\nabla)\bar{V}$. One of the advantages of VSH is that products of VSH or SSH are reduced to sums. This property is desirable both analytically (because it simplifies the equations as a rule) and computationally, because it is possible to consider

aliasing effects. What we require, if not an explicit product rule, is at least the knowledge of the harmonic content of the product of candidate members. Thus, for example, in a Fourier sine series the product of any two terms is

$$A_m \sin mx \cdot B_n \sin nx = \frac{A_m B_n}{2} \left(-\cos(m+n)x + \cos(m-n)x \right) \tag{2-3}$$

Thus, the harmonic content of the product of two sine series is the sum of the harmonic content of each series. Similarly, the product of two polynomials of degree K is a polynomial of degree 2K.

V.2.4 Applicability

This criterion is difficult to define. Some function sets which meet all the criteria - orthogonality, differentiability closure, product rule, etc., are not suitable candidate sets for reasons not immediately clear in the light of previous discussions. For example, the fact that Fourier series are periodic causes problems when these functions are used. One could add a criterion of non-periodicity but this seems too sweeping. The spherical Bessel functions are more applicable than Fourier Series even though the Bessel Functions do not meet all the above criteria. (They lack a product rule.)

The main criterion in deciding applicability is practical experience, coupled with analysis based on analytic work on linear problems (see previous subsection), or on fits to real data, or both. A third consideration is programming difficulty and evaluation time, although in the current system most of the data needed for

fitting is precomputed, and the actual fitting reduced to vector-matrix multiplications. Since nonlinear calculations are carried out at grid points using the transform method, the actual mathematical formulation of the product rules, D₋ operator rules, etc., is not as important as the knowledge of the harmonic content of the formulas, from a computational point of view.

V.3 Results of Radial Function Studies

V.3.1 Summary

A number of function sets has been investigated to date. A complete statement of the results would be extremely tedious, since it would involve statements of the pertinent properties of the functions in question. In all cases, the functions are well known and documented elsewhere, notably references (1), (2), (3). The policy adopted, therefore, is to provide a summary statement of the virtues (or otherwise) of the function sets in the light of sub-section V.2, and to comment briefly on the summary if necessary. Table V.1 provides the summary and the subsequent discussion provides the commentary.

Functions	Orthogonal	Derivative Closure	Product Rule	Applicability
Powers of r	No	Yes	Yes	Diverge at $r = \infty$
Powers of 1/r	No	See (1)	Yes	Bad choice based on fits to data
Fourier Series	Yes	Yes (2)	Yes	Problems with periodicity at end-points
Legendre Polynomials	Yes	Yes	Yes	Current candidates problems with mapping interval of orthogonality
Spherical Bessel Functions	Yes (3)	Yes	No	Best choice except for lack of product
Associated Laguerre Functions	Yes			Currently being investigated.

Table V.1 Radial Function Summary

NOTES:

(1) If one truncates, then terms on the derivatives of inverse powers of r will be lost.

(2) Closed under $\partial/\partial r$; the operator $(\partial/\partial r + 1/r)$ introduces some problems.

(3) Orthogonality interval is very cumbersome.

(4) Actually, radial Eigenfunctions of the one-electron atom.

V.3.1 Powers of r, and Inverse Powers of r or 1/r

Powers of r, i.e., a polynomial in r, have the advantage of being easy to compute. However, they are divergent for large r. Fits to data indicate that these functions are not particularly applicable. Static linear problems sometimes yield powers of r as radial eigenfunctions. For example, the solution of the equation

$$\nabla^2 \varphi(r,\theta,\varphi) = 0 \tag{3-1}$$

(Laplace's equation) can be written as

$$\varphi(r,\theta,\varphi) = \sum \left(A_L r^L + B_L r^{-(L+1)} \right) Y_L^M(\theta,\varphi) \tag{3-2}$$

with A_L and B_L constants. However, no dynamic problem (i.e., one involving $\partial/\partial t$ of any variable) thus far attempted has yielded powers of r or inverse powers of r as a solution.

V.3.2 Fourier Series

Fourier series have been used to fit data with variable results. In some cases good fits are obtained, but in others there are problems. These problems arise from the fact that the Fourier functions are periodic but the atmosphere is not. Any fitting process must make some assumptions as to how the data will be repeated outside the interval of the atmosphere.

As an example, one can assume the data repeat symmetrically with odd functions predominating, in which case a sine series is obtained. In each case, however, the data must be examined to determine the proper assumptions, or else the series may not fit well over the prime interval. The resulting complications have led us to discard Fourier series as radial functions, at least for the present. Otherwise, Fourier Series are a good choice insofar as the criteria of the previous section are concerned.

V.3.3 Legendre Polynomials

These are the choice of at least one other spectral modeling group (4) (5). As is well known, the Legendre polynomials are an orthogonal set of polynomials, with derivative closure and well-defined product rule. Computationally they are very well behaved.

The main drawbacks of Legendre Polynomials are that (1) they are unsuitable for variables that tend to zero at the upper atmosphere, such as pressure; and (2) they do not arise in any linearized hydrodynamic problem considered to date, and therefore are probably not "natural" in some sense.

In spite of these drawbacks they are being used as radial functions in the model described in section III.5.

V.3 4 Spherical Bessel Functions

Spherical Bessel Functions are in many respects the most "natural" choice for radial functions. They are in fact Bessel functions of half-integer order, and arise in the solution of many linear problems. For example, the sound wave equations (section II.4), equation (1-1) in this section, and Schrödinger's equation in quantum mechanics for the Eigenfunctions of a particle in a "spherical square well" potential all lead to Spherical Bessel functions for the radial dependence.

With regard to the D operator, it can be shown from the recursion relations in ref. (1) that, if $f_n(r)$ is a spherical Bessel function of order n, then

$$D_L f_n(r) = \frac{L+n+1}{2n+1} f_{n-1}(r) + \frac{L-n-1}{2n+1} f_{n+1}(r) \qquad (3\text{-}3)$$

The simplicity of this result does not extend over into products. Thus far, it has not been possible to derive a product rule, much less to show that the set is closed under products. Products appear to (a) double the argument of

the function, and (b) introduce a polynomial in 1/r which is not a Spherical Bessel function.

With regard to orthogonality, the Bessel functions are orthogonal over zeros of adjacent functions, which introduces a computational problem.

The literature on these functions is not extensive, which has also delayed their evaluation.

For all these reasons, Spherical Bessel functions remain a potential choice but have not been used to date in implementations.

V.3.5 Associated Laguerre Functions

To be exact, we refer to the radial eigenfunctions of the one-electron atom (6). These arise in the solution of Schrödinger's equations for a particle in a 1/r potential field. These functions are currently under investigation.

References for Section V

(1) Abramowitz & Stegun, Handbook of Mathematical Functions. Dover, N.Y. 1965.

(2) Jahnke, E. and Emde, F. Tables of Functions. Dover, N.Y. 1943

(3) Erdelyi et al. Higher Transcendental Functions. McGraw-Hill, N.Y. 1955.

(4) Elissen, E., Machenhauer, B. and Rasmussen, E. On a Numerical Method for Integration of the Hydrodynamic Equations, Etc. Copenhagen Institute for Theoretical Meteorology, Report No. 3 (1972)

(5) Machenhauer, B. and Rasmussen, E. On the Integration of the Spectral Hydrodynamical Equations with a Spectral Method. Copenhagen Institute for Theoretical Meteorology, Report No. 4 (1972)

(6) Bethe, H. A. and Salpeter, E. E. Quantum Mechanics of One-and-Two Electron Atoms. Academic Press, N.Y. (1957)

VI. Data Handling

This section is a report on numerical work, fitting published data [1][2] with vector spherical harmonics.

The results of the fits provide information on the number of terms needed to provide accuracy, on the radial behavior of the atmospheric fields, and on computational problems that will be encountered. The principal results encountered are summarized below.

- Number of Terms. In terms of the index L in the scalar or vector spherical harmonics, it appears that L=10 to L=20 is sufficient to fit the published data, with ample accuracy. Since most published data are zonally averaged, it is difficult to make similar claims for the index M. (The index L determines the maximum frequency in θ , the index M the maximum frequency in φ).

- Radial Behavior. No overall conclusion to be drawn. In general, there appear to be periodicities combined with overall trends (e.g., pressure approaches zero). The radial behavior can be duplicated with, for example, Legendre Polynomials or Fourier Series (see Section V).

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(1) Schutz, C. and Gates, W.L., Global Climatic Data for Surface, 800 mb, 400 mb, January. Report No. R-915-ARPA, the Rand Corp., Santa Monica, CA (Nov. 1971).

(2) Oort, A.H., and Rasmusson, E.M., Atmospheric Circulation Statistics, NOAA professional paper 5. U. S. Dept. of Commerce, NOAA, Rockville, MD (Sept. 1971).

- Computational Problems. No major computational problems arise on the fitting of data, except those connected with aliasing (see section IV). In the case of using Fourier Series for radial representation, additional problems are encountered (section V). All computations presented herein were performed in APL without any difficulty. When fitting data it is advantageous to use a grid spacing of "gaussian" points to perform the necessary quadratures. Thus, it is necessary to reduce the data to such a grid prior to fitting. In practice this could present some difficulties. As with all data reduction schemes, data gaps pose a problem.

The remainder of this section consists of tabulated results.

ZONALLY AVERAGED MEAN GEOPOTENTIAL HEIGHT (2)

1000 to 700 MB

$$r = \sum r_L^0(p)\, Y_L^0(\theta,\varphi)$$

L	1000 MB	950 MB	900 MB	850	700
0	381.46	1980.25	3537.34	5242.06	10770.57
1	-68.83	31.01	-121.97	-112.71	-422.05
2	-103.76	105.33	2.88	117.58	54.41
3	8.02	236.38	126.00	258.86	191.66
4	-70.57	167.78	57.59	191.81	103.86
5	-79.98	152.82	51.50	178.70	101.70
6	-53.38	157.15	61.39	174.33	97.24
7	-47.81	131.80	49.30	148.70	89.33
8	-36.48	111.79	43.28	122.26	71.87
9	-30.54	83.73	34.32	94.67	53.65
10	-21.54	61.35	25.93	69.84	40.91
11	-14.40	39.02	15.68	42.93	24.44
12	-5.06	25.08	10.79	25.96	15.59
13	-1.79	11.19	5.88	12.39	6.69
14	-0.50	5.26	2.80	4.95	2.00

ZONALLY AVERAGED MEAN GEOPOTENTIAL HEIGHT (2)

500 to 100 MB

L	500 MB	400 MB	300 MB	200 MB	100 MB
0	20004.17	25770.07	32942.38	42353.49	57588.72
1	-788.91	-1119.09	-1307.08	-1532.12	-1088.19
2	102.83	41.00	253.29	317.51	322.74
3	242.39	100.92	373.29	397.74	485.47
4	95.95	17.92	159.92	155.26	349.25
5	123.05	34.52	240.17	275.85	405.95
6	110.93	26.16	205.31	234.15	368.15
7	101.58	28.94	173.21	180.00	300.33
8	83.05	23.99	153.16	167.92	253.92
9	59.56	13.66	112.02	124.16	197.47
10	47.91	16.45	84.77	89.68	139.35
11	27.21	6.15	50.03	54.65	84.54
12	18.43	7.93	32.66	36.[illegible]2	49.53
13	6.91	1.80	14.49	16.36	19.53
14	3.17	1.98	5.36	6.51	5.15

ZONALLY AVERAGED MEAN JANUARY

ZONAL GEOSTROPHIC WIND [(1)]

$$\bar{V} = \sum_L C_L^0(p) Y_L^M(\theta, \varphi)$$

L	800 MB	400 MB
0	0.00	0.00
1	-6.44	-35.51
2	1.33	5.17
3	-15.05	-29.67
4	2.60	11.29
5	6.35	19.98
6	-1.69	-4.62
7	5.25	-2.49
8	-4.67	-6.66
9	-1.60	-0.76
10	1.68	3.44
11	-0.68	-1.22
12	0.22	0.75
13	-1.04	-1.87
14	0.38	-0.02
15	0.38	0.85
16	0.00	-0.76
17	-0.07	-0.56
18	-0.28	0.11
19	0.07	0.36

ZONALLY AVERAGED MEAN
JANUARY TEMPERATURE (1)

$$T = \sum_{L} T_L^0(p) Y_L^0(\vartheta, \varphi)$$

L	1000 MB	800 MB	400 MB
0	43.33	14.65	-98.94
1	-18.20	-12.18	9.82
2	-51.70	-39.09	-32.37
3	0.46	1.22	1.31
4	-5.64	-0.88	1.32
5	4.01	2.32	1.90
6	-0.70	1.72	1.57
7	1.10	1.37	1.33
8	-3.10	-1.61	-1.44
9	2.12	0.26	-0.03
10	-1.07	-0.17	0.08
11	0.40	0.05	0.23
12	-0.06	-0.22	-0.11
13	-0.67	-0.15	0.12
14	0.19	0.29	-0.09
15	-0.91	-0.07	-0.22
16	-0.20	-0.02	0.00
17	-1.10	-0.11	-0.01
18	-0.27	-0.09	0.06
19	-0.25	-0.14	0.02

VII Program System Development

VII.1 Work to Date

Program modules have been developed to perform all the functions necessary to apply VSH to partial differential equations expressed in spherical coordinates. These include modules to:

- o Numerically fit spectral data to obtain estimates of radial dependence
- o Form radial differential operators in spectral space for use in forming three-dimensional spatial differential operators
- o Form three-dimensional spatial differential operators - curl, divergence, etc. -- in spectral space
- o Form horizontal spatial differential operators
- o Transform between spectral and physical space
- o Perform time and space integrations in spectral space
- o Perform numerical quadrature in spectral space
- o Generate tabular data for the transformation, fitting and quadrature modules
- o Generate the job control language procedure to execute the programs and for describing data sets to be used by them

In addition, data organization, both for core and peripheral storage, has been developed to support and inform these program modules. Finally, programs to control the sequence of execution of the program modules and to form the model described in Section 3.7 have been implemented.

Programs have been developed in terms of the general system described in the previous report. Capabilities not described in that design description, but required for the Mintz-Arakawa implementation, such as spatial integration and numerical quaduráture, have also been implemented. In all, some 9,000 lines of code have been generated since January of 1973.

The equations described in Section 3.6 and implemented as in Section 3.7 have been programmed. Debug runs are being made on an IBM System/370 Model 155 computer.

VII.2 Lessons Learned

It is somewhat ironic that a program system that was designed with the intention of anticipating change should find it difficult to accommodate certain functions not clearly foreseen at the time of the design. But work on the Mintz-Arakawa implementation has pointed up several areas where rethinking and reworking are in order.

The problem of aliasing when dealing with transformations between spectral and physical domains was not fully understood at the time of the original design. Truncations of spectral expansions over theta and phi were done to avoid aliasing, but the need for a consistent radial truncation was not appreciated.

It was the original intent of the system design to isolate the model in physical space. The equations would be formed and boundary conditions applied in one compact and related set of program modules. There are, however, instances where portions of the model might more efficiently be formed in spectral space. Certain boundary conditions are more readily cast and applied in the spectral domain. So rather than isolate a given model, it may be more advantageous, even from a research view, to distribute its elements through the system.

The system was originally conceived for solving temporal differential equations. The final cast of our Mintz-Arakawa model, involving differential equations in τ and σ has made it obvious that spatial differential equations must be accommodated. Also, while specific integration algorithms were not considered during the design, the need for sequential rather than simultaneous integration schemes has made the control flow for integration more cumbersome and less straightforward than originally thought.

All of these considerations can be fit into the current system. With the exception of forming model equations in spectral space, they all have been for the Mintz-Arakawa implementation. These changes have not, however, been made with the generality and coherence that the original design strove for. The key to the latter is anticipation of functional requirements and careful planning for their needs. This is particularly true for data organization. The heart of the current system is the organization and content of its data storage areas. Many of the above functions, while implemented in the system, use data areas originally designed for other functions. This is clumsy and at times restrictive. If the system is to accommodate the newly appreciated range of capabilities with the originally intended generality, ease of use and case of change, a design review is needed. This is particularly true in the areas of data needs and organization, and high level functional control flow.